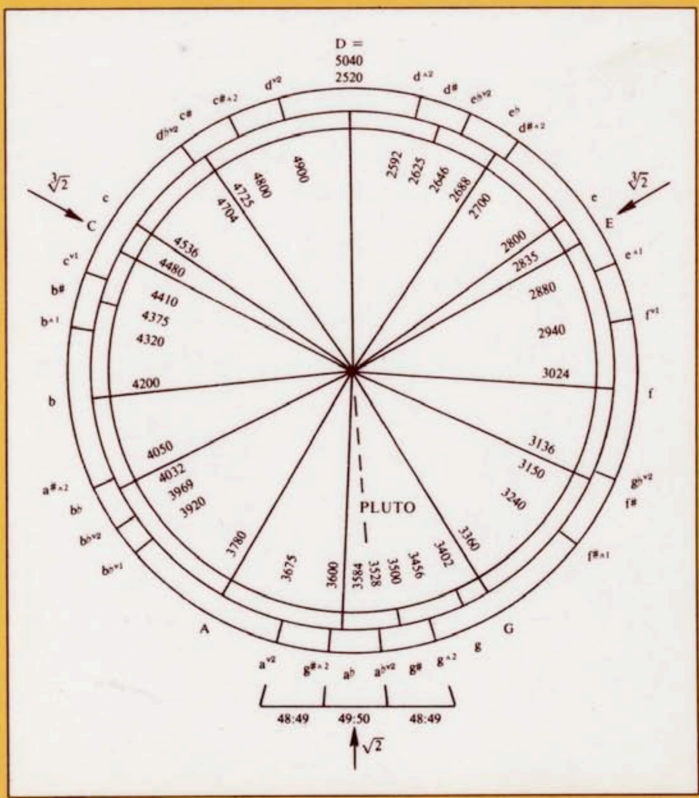
The

Pythagorean Plato

Prelude to the Song Itself



Ernest G. McClain

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to Ernst Levy
musician, philosopher, friend

The consideration of which numbers are concordant and which not, and why in each case . . . don't we know that all of this is a prelude to the song itself, . .. the song itself that dialectic performs?

(Republic 531c-532a)

Contents

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Plato: The Laws, *translated by Trevor J. Saunders, © Trevor J. Saunders, 1970, Penguin Classics, Harmondsworth.*

Timaeus and Critias, *A. E. Taylor, Methuen and Co. Ltd., London, 1929.*

Plato's Cosmology, *Francis M. Cornford, Humanities Press Inc., New Jersey, and Routledge & Kegan Paul, Ltd., London, 1937.*

Robert J. Brumbaugh has kindly consented to the reproduction of his maps of Atlantis and Magnesia (cf. figs. 32 and 57) from his work, *Plato's Mathematical Imagination,* (Bloomington: Indiana University Press, 1954, and New York: Kraus Reprint Corporation, 1968).

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Interdisciplina, *"The Tyrant's Allegory," vol. 1, no. 3, Spring, 1976, revised here as chapter 3.*

The Mathematics Teacher, *"Pythagorean Paper Folding: A Study in Tuning and Temperament," vol. 63, no. 3, March 1970, revised here as appendix IV.*

Charts and Tables

1 Introduction

THE PROBLEM

Plato's later dialogues abound in mathematical allegories. *Timaeus* begins with a very long one, *Statesman* contains a short one, the *Republic* has three, and both *Critias* and *Laws* are permeated with them from beginning to end. When Plato died in 347 B.C. his pupils and friends immediately began to argue about these mathematical constructions and about Plato's purpose in using them for models of souls, cities, and the planetary system. By the beginning of the Christian era much of Plato's mathematics had become a riddle. Many rivals clamored for recognition as the "single harmony" Socrates heard from the planets.[1] A certain number which he confidently proclaimed "sovereign" in political theory was labelled "numero Platonis obscurius" by Cicero (c. 100 A.D.), with the hearty concurrence of later scholars; an interpretation which Nicomachus promised at about this time was either lost or never written.[2] By the fifth century A.D., Proclus, one of the last to head the Platonic Academy, could not pretend to understand Plato's arithmetic, although he was astute enough to label as spurious a then popular interpretation of the *Timaeus* "World-Soul."

Down through history Plato's mathematical allegories defied Platonists either to reconstruct his arithmetic or to find in it the implications he claimed for it. The twentieth century opened with a vigorous new effort. In 1901 James Adam searched for the common-sense implications of the allegories of the *Republic,* rejecting as spurious most of the interpretation of the intervening millenia, and setting a new standard of rigor in remaining respectful of Plato's precise words.[3] In 1928 A.E. Taylor performed the same service for *Timaeus,* gleaning from his effort one principle which—had he known how to apply it—might have saved him from the failure which plagued his predecessors: Plato, he wrote, "demands perfect symmetry."[4] In 1937 Francis Cornford, perhaps the most prestigious of 20th century Platonists, inadvertently took a step backwards. Cornford concluded that the difficulties which arise in abstracting a planetary system from Plato's musical arithmetic in

Timaeus are due to a metal "armillary sphere" which the Academy possessed (and an unworkable one, at that): "Plato probably had it before him as he wrote."[5] In 1945, in his translation of the *Republic,* Cornford not only omitted "the extremely obscure description" of Socrates' "sovereign number," but he also allowed himself to "simplify the text" of the tyrant's allegory.[6] Then, in 1954, Robert Brumbaugh opened a whole new vista on Plato's mathematics.

In *Plato's Mathematical Imagination* Robert Brumbaugh collected every Plato text concerned with mathematics and subjected this whole corpus to a rigorous logical analysis.[7] Brumbaugh reconstructed very many of the diagrams to which Plato's texts allude and without which they seem obscure. He noted that the principle of "aesthetic economy" in Pythagorean use of smallest integers for examples of general relations in number theory—is itself a purely logical device in an age which had not yet developed a general notation for algebraic variables.[8] He noted the importance for Plato of the circle as *cyclic* metaphor involving "some sort of *reciprocity."*[9] He abstracted the general principles in Plato's use of algebraic and geometric metaphor while showing that none of the extended allegories had ever been given a satisfactory detailed interpretation. From studies of Platonic arithmetic—described by Aristotle as involving "a Dyad of the Great and the Small" Brumbaugh arrived at a novel suggestion: "The concept of a modulus, a mathematical metaphor that is natural for reiteration, seems to have been already formulated by Plato's time."[10] The "modulus" relevant to Plato, he added—"extremely attractive as a starting point for interpretations of...the problematic dyad—is that based on the 2:1 ratio of the musical octave.

A MUSICAL POINT OF VIEW

Either unknown to Platonists or ignored by them, a truly musical effort to expound Plato's mathematical meaning has been under way for a century. In 1868-76 Albert von Thimus had already anticipated brilliantly the musical implications of Taylor's principle of "perfect symmetry" and Brumbaugh's principle of "cyclic reciprocity" applied to the *Timaeus* creation myth.[11] Von Thimus' ideas were expounded by Hans Kayser and Ernst Levy, and developed more fully in Levy's treatises on harmony (in French in 1940 and English in 1950, but still unpublished) and in his essays and lectures on Pythagoreanism.[12] It was my own good fortune to have been a colleague of Ernst Levy for several years and to have inherited from his fund of priceless insights into

Pythagoreanism the particular insight that the formula for Socrates' seemingly impenetrable "sovereign" political number defines what musicians know as Just tuning—idealized, but impracticable on account of its endless complexity—and that the political disaster Socrates foretold must therefore be related to the difficulty musicians have always known. The often-expressed conviction of Hugo Kauder, our mutual friend, that Pythagorean tuning functioned as a kind of "temperament" for the Greeks, was a prescience which has proved helpful in developing Levy's intuition.

My work progressed by applying the von Thimus-Levy musical insights to the mathematical foundation provided by Adam, Taylor, and Brumbaugh, under the guidance of Siegmund Levarie—mentor, friend, and colleague—who has participated in my studies from the beginning. The result is a thesis which none of us could have anticipated: not only are all of Plato's mathematical allegories capable of a musical analysis—one which makes sense out of every step in his arithmetic—but all of his allegories taken together prove to be a unified treatise on the musical scale so that each one throws light on the others. However, it is perhaps even more remarkable that, when the *Republic, Timaeus, Critias,* and *Laws* are studied as a group as a unity, it then proves possible to explain virtually every Platonic mathematical riddle with help from related passages, that is, in Plato's own words.

For possibly thousands of years before Plato, music provided a meaningful correspondence between number and tone in the readily observed and easily measured correlations between string lengths (on Hindu-Sumerian-Babylonian harps) and the tonal intervals they sound. Other factors being equal, halving a string length changes its pitch by an octave; the ratio 1:2 thus corresponds to an acoustical experience—and a most important one, for every such division by 2 is perceived as a kind of "cyclic identity" and is assigned the same letter name in modern notation. Subdivisions of this octave space can be defined as smaller ratios (i.e., between successively larger numbers). Although the ear cannot verify results with any accuracy beyond the first few subdivisions of the "fifth" 2:3, "fourth" 3:4, and perhaps the "major third" 4:5 and "minor third" 5:6, yet even micro-intervals can be readily calculated by he number theorist, to whatever limits please him, by continued operations with these same first six integers. And the tuner can follow, generating each tone from the last one, never daring to omit intermediate steps.

The very simplicity of tuning theory probably accounts for its being the first physical science to become fully mathematized. Although Plato obviously knew the monochord—an instrument designed to keep all other factors constant while string length alone is varied—he insisted that tuning theory be studied exclusively in mathematical terms. He writes amusingly of men who "harass the strings and put them to the torture, racking them on pegs," then setting their ears alongside "as though they were hunting a voice from the neighbor's house," while arguing with each other about "the smallest interval by which the rest must be measured, while others insist that it is like those already sounded." Such men, he complains, "put ears before the intelligence."

> *They seek the numbers in these heard accords and don't rise to problems, to the consideration of which numbers are concordant and which not, and why in each case.*
>
> *(Republic 53 la.c)*

Plato's strong language requires that we too "rise to problems" and think about numbers themselves and avoid deceiving ourselves that we can solve his problems simply by charting the tonal meanings of numbers along a monochord string. I will introduce schematic monochords in a few places, but mainly to render vivid to the reader the contradictions which arise when we ignore Plato's ridicule of that method.

Musicians who build and tune their own instruments know very well that the octave 1:2 cannot be subdivided equally by the "pure" musical ratios of rational numbers like those mentioned above. Today we arbitrarily subdivide the octave space into twelve equal parts so that a "semitone" of $1/12$th of the octave has the numerical value $\sqrt[12]{2}$, and every larger interval is some multiple number of semitones, the smallest interval in the system. The integer ratios Plato knew are either slightly too large or too small to generate such a "closed" system; they lead, in fact, toward the musical chaos of an infinite number of tones.

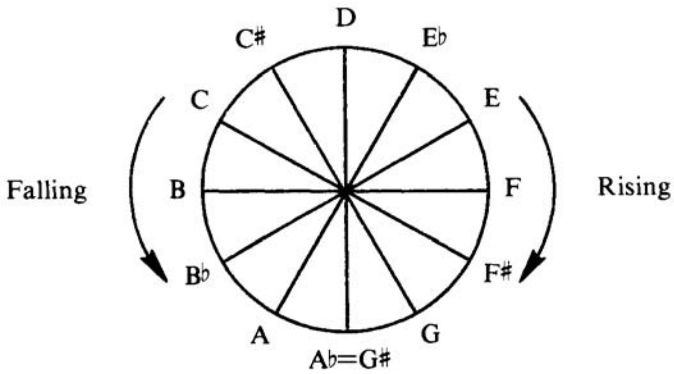


FIGURE 1

The Equal Tempered Scale

intiation of the problems which Plato dramatizes can be seen in the fact that powers of 2 (4, 8, etc.) defining octaves are *even* numbers which never coincide with powers of 3 (9, 27, 81, etc.) defining fifths and fourths by *odd* numbers, and neither of these series agrees with powers of 5 (25, 125, etc.) defining musical thirds. Tuning a 12-tone system is something of an art, then, for we must orient ourselves by the few "pure" intervals which we can hear accurately and then slightly deform them—we call it "tempering"—to ensure cyclic agreement.

From a musician's perspective, Plato's *Republic* embodies a treatise on equal temperament. Temperament is a fundamental musical problem arising from the incommensurability of musical thirds, fifths, and octaves. The marriage allegory dramatizes the discrepancy between musical fifths and thirds as a genetic problem between children fathered by 3 and those fathered by 5. The tyrant's allegory dramatizes the discrepancy between fifths and octaves as that between powers of 3 and powers of 2. The myth of Er closes the *Republic* with the description of how the celestial harmony sung by the Sirens is actually tempered by the Fates, Lachesis, Clotho and Atropos, who must interfere with planetary orbits defined by integers in order to keep them perfectly coordinated. In Plato's ideal city, which the planets model, justice does not mean giving each man (men being symbolized by integers) "exactly what he is owed," but rather moderating such demands in the interests of "what is best for the city" (412e). By the 16th century A.D., the new triadic style and the concomitant development of fretted and keyboard instruments transformed Plato's theoretical problems into pressing practical ones for musicians and instrument makers. With the adoption of equal temperament about the time of Bach we made into fact what for Plato had been merely theory. Musically the *Republic* was exactly two thousand years ahead of time.

In the pages which follow I shall suppress most of my own debts to von Thimus, Levy, Adam, Taylor, and Brumbaugh among modern scholars, and to Aristotle, Plutarch, Ptolemy, Nicomachus, and Proclus among the ancient ones, in order to let the reader enjoy the experience of learning Platonic arithmetic directly from Plato. He speaks in the archaic language of pre-Euclidean mathematics. His musical puns are incessant and often untranslatable.[13] There is extreme economy in his formulas. He loves metaphor with multiple overtones of meaning. He fuses comedy with "tragedy" (meaning high seriousness) exactly as he suggests at the end of the *Symposium* (223d). His words glow with a vitality none can match. I may trespass here on his intent never to put into words a formal exposition of a subject to which he devoted many years of his life, and agree wholeheartedly with him in his claim that, "if there were to be a treatise or lecture on this subject, I could do it best" *(Letter* 7, 341c-d).

He was referring to his own philosophy, not musical mathematics; his explanations of either would have been "best."

That Plato was accustomed to addressing an audience already familiar with the simple materials of Greek musical theory will be evident. I shall limit musical examples to the minimum needed to understand his arithmetic, promising the reader in advance that only one musical octave is involved and that the only mathematical complexities are those which emerge from multiplication and division by the first ten integers. The large numbers which appear are merely the "octave-doubles" which allow us to avoid fractions; generative ratios use small numbers.

That the allegories are musical and that they involve one octave, ten numbers, and the principle of "reciprocity," we can learn from Plato himself.

PLATO'S MUSICAL THESES

For the soul: Education in music is "most sovereign" *(Republic* 401d). The just man "will always be seen adjusting the body's harmony for the sake of the accord in the soul" *(Republic* 591d). "Argument mixed with music . . . alone, when it is present, dwells within the one possessing it as a savior of virtue throughout life" *(Republic* 549b).

For the city: The city's guardians must "build their guardhouse" in music and permit "no innovations," for "never are the ways of music moved without the greatest political laws being moved" *(Republic* 424c,d). "Moderation" will stretch through the whole city as it does "from top to bottom of the entire scale, making the weaker, the stronger, and those in the middle . . . sing the same chant together" *(Republic* 432a). "Our songs have become laws," a pun on nomoi as meaning both laws and traditional melodies for the recitation of the classics *(Laws* 799d).

For the heavens: The Sirens mounted on the rims of our "planetary whorls" each sing one sound, one note, so that "from all eight is produced the accord of a single harmony" *(Republic* 617b).

For the unity of his mathematical allegories:

> *I think it better, my good friend, that my lyre should be discordant and out of tune, and any chorus I might train, and that the majority of mankind should disagree with and oppose me, rather than that I, who am but one man, should be out of tune with and contradict myself*
>
> *(Symposium* 482c)

e) For restriction to *one* model octave: Pattern is the Living Being that is forever existent" *(Timaeus* 37c,d). "It was therefore, for the sake of a pattern, that we were seeking both for what justice by itself is like, and for the perfectly just man" *(Republic* 472c). "A city could never be happy otherwise than by . . . (imitating) the divine pattern" *(Republic* 500e). God, the maker of patterns, "whether because he so willed or because some compulsion was laid upon him not to make more than one...created one only" *(Republic* 597c). For musical theory, all octaves have the *same* pattern. For Aristotle, the octave was the prime example of a Platonic *form.* [14]

f) For relating tone to number, and both to geometry:

> *To the man who pursues his studies in the proper way, all geometric constructions, all systems of numbers, all duly constituted melodic progressions, the single ordered scheme of all celestial revolutions, should disclose themselves... [by] the revelation of a single bond of natural interconnection.*
>
> *(Epinomis* 991e, 992a)

g) For restricting *number* to *integers* and founding political theory on the first ten: "As for those children of yours . . . , I don't suppose that while they are as irrational as lines you would let them rule in the city..." *(Republic* 534d)? When "true philosophers" come to power, "all those in the city who happen to be older than ten they will send out to the country" *(Republic* 541a). Fractions must always be avoided by the use of least common denominators, according to the principle enunciated as follows:

> *For surely, you know the way of men who are clever in these things. If in the argument someone attempts to cut the one itself [i.e. use a fraction], they laugh and won't permit it. If you try to break it up into small coin, they multiply, .. .*
>
> *(Republic* 525d)

h) For the rigorous application of reciprocity, or the study of opposites:

> *Some things are apt to summon thought, while others are not, ... Apt to summon it [are] those that strike the sense at the time as their opposites.*
>
> *(Republic* 524d)

Platonists have always known that in Pythagorean ratio theory numbers function reciprocally as both *multiples* and *submultiples* of some basic unit (n and 1/n). The secret to Platonic mathematical riddles is that we must study *reciprocals,* compressed to one model octave. The game with reciprocals illustrates a theory of perception: *qualities* depend on sensation, which depend in turn upon a theory of "flesh, or the mortal part of the soul," a dilemma which requires that we "assume the existence of sensation . . . and afterward turn back to examine what we have assumed" *(Timaeus* 61c,d). It is this turning back to criticize one's initial assumptions which separates Plato from all philosophy developed from "first principles," as Western philosophy tried repeatedly from Aristotle onwards. No assumption we can make about Plato's tone-numbers makes any sense until we have "turned back to study them also from the opposite point of view.

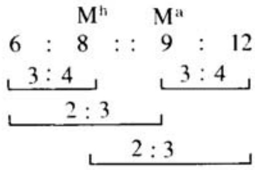
We turn now to *Epinomis* and *Timaeus* for Plato's exposition of number theory.

PLATO'S TONAL THEORY OF NUMBERS

Greek musical theory is founded on the so-called "musical proportion" 6:8::9:12 which Pythagoras reputedly brought home from Babylon. It is this proportion which exemplifies the science Plato labels *stereometry* (the gauging of solids),

> *a device of God's contriving which breeds amazement in those who fix their gaze on it and consider how universal nature molds form and type by the constant revolution of potency and its converse about the double in the various progressions. The first example of this ratio of the double in the advancing number series is that of 1:2.*
>
> *(Epinomis* 990e)

Every arithmetical "double" defines a musical octave 1:2, and the proscription against fractions ("dividing the one") requires a new "double" for every change in perspective. (Notice the claim that "universal nature molds form and type" on the model of the musical octave-double, words which support Brumbaugh's reference to a "modulus of doubles.") Within this octave space the arithmetic mean ($M^a = 1\frac{1}{2}$) and its "sub-contrary" or "harmonic" derivative mean ($M^h = 1\frac{1}{3}$) require 6 as *least common denominator* and thus appear to a Pythagorean as, respectively, 9 and 8:



These ratios define the only *fixed tones* in Pythagorean tuning theory and appear as the *invariant* element in every Platonic mathematical allegory. *Epinomis* continues with a discussion of the "double" and a definition of the means which is worth quoting in detail, whether written by Plato himself or by his scribe, Philip, as some contend; its extravagant praise of the insight to be gained from this example is in due proportion to the importance it plays in Platonic cosmology and political theory.

> *The first example of this ratio of the double in the advancing number series is that of 1 to 2; double of this is the ratio of their second powers (1:4), and double of this again the advance to the solid and tangible, as we proceed from 1 to 8 (1,2, 2^2,2^3); the advance to a mean of the double, that mean which is equidistant from lesser and greater term (the arithmetical), or the other mean (the harmonic) which exceeds the one term and is itself exceeded by the other by the same fraction of the respective terms these ratios 3:2 and 4:3 will be found as means between 6 and 12—why, in the potency of the mean between these terms (6,12), with its double sense, we have a gift from the blessed choir of the Muses to which mankind owes the boon of the play of consonance and measure, with all they contribute to rhythm and melody (991).*

This tribute to the musical gift of the Muses must not distract musicians from recognizing that, for Plato, sight—not hearing—"is the cause of the highest benefits to us" *(Timaeus* 47a). He speaks of "those who fix their gaze" on his example—"the constant revolution of potency and its converse" using words which warn us that number and tone must be translated into geometric imagery, and revealing that his own primary image is the *circle,* purest embodiment of the notion of cycle. Platonic cities are circular; their models are "tone-circles." "The regime, once well started," Socrates claims, "will roll on like a circle in its growth"

(Republic 424a). The celestial city revolves before our eyes: "The sight of day and night, of months and the revolving years, of equinox and solstice, has caused the invention of number . . . whence we have derived all philosophy" *(Timaeus* 47b). The "worst" city Plato could invent—Atlantis—is a ring of four concentric circles; and his "practicable" city of Magnesia expounded in *Laws* is a double ring divided "into twelve sections by lines radiating from (its) central point," (745) like the twelve tones of our chromatic scale (see fig. 1). Socrates discovers the "justice" aimed at in the *Republic,* "rolling around at our feet from the beginning" (432d), a possible allusion to Plato's own circular designs in the sand.

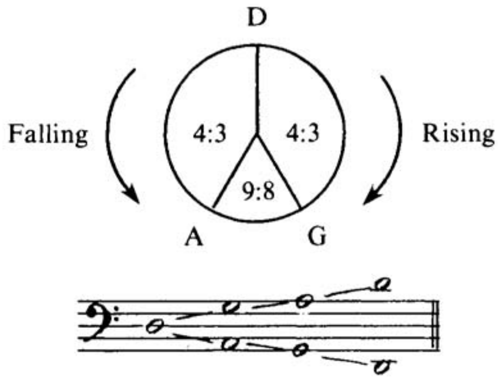
To understand why the musical proportion 6:8::9:12 is itself visually misleading, note that these numbers function in reciprocal ways (as ratios of wave-length and of frequency, or as multiples and as submultiples of some unit of string length), and hence they apply to *both* rising and falling sequences of pitch.

			M^h		M^a		
	6	:	8	: :	9	:	12
rising	D		G		A		D
falling	D		A		G		D

In the octave on D, for instance, the tones A and G reverse their roles as arithmetic and harmonic means; both tones and numbers, then, delude us with mere "appearances." Notice how the geometry of the "tone circle" helps us rise to a higher insight, hence closer to the "reality" of *invariant truth.* The legislator, Plato writes, must

> *assume it as a general rule that numerical division in all its variety can be usefully applied to every field of conduct. It may be limited to the complexities of arithmetic itself, or extended to the subtleties of plane and solid geometry; it is also relevant to sound and to motion, straight up or down or revolution in a circle.*
>
> *(Laws* 747a)

In the circle beginning and end coincide so that the reference tone, D, actually *functions visually* as *geometric* mean between the symmetrically located arithmetic and harmonic means—whose own positions alternate according to whether numbers are thought of as multiples or submultiples, and attached to tones which rise or fall. By this device we have acquired a Socratic "seat in the mean" *(Republic* 619a) which never changes, for the tone-field will develop symmetrically to the right



Rising	D		G		A		D
Falling	D		A		G		D
Ratios	6	:	8	: :	9	:	12

FIGURE 2

Circular Projection of the Musical Proportion

and left of D (used here because it happens to be the center of symmetry in modern alphabetical notation)—provided, that is, we remember to graph the *dialectical tonal meanings* of every arithmetic formula Plato mentions. Our reference tone, D, means actually “1” *functioning* as *geometric mean* between each rational number and its reciprocal, hence symbolizing “God” by its absolute invariance. Every number (meaning *integer,* in Plato's time), can be conceived as a “multiplicity of 1’s” hence also, metaphorically, as a mask of God,” to borrow Joseph Campbell's term. The various numbers to which Plato directs our attention and which we shall interpret as D are merely transformations of 1 = geometric mean; such transformations are required to avoid fractions while maintaining reciprocity, two basic conditions every construction must satisfy. Multiplication by any positive integer preserves the relevant formal properties of the rational numbers. In particular, the process of “clearing fractions” so fruitful from a number-theoretic standpoint—preserves the property of being a “mean.” Everything in Plato's mathematical world is held together by the three means—arithmetic, harmonic, and geometric.

> *And of all bonds the best is that which makes itself and the terms it connects a unity in the fullest sense; and it is of the nature of a continued geometrical proportion to effect this most perfectly.*
>
> *(Timaeus* 31c)

Now, visually, we see that A:D::D:G, just as we recognize aurally the invariance of the "perfect fourths" which they frame when sounded in rising order, and the "perfect fifths" which they sound in falling order. In the tone-circle we escape time and change, in a frozen image which generates, paradoxically, an intense mental activity in the mind of the beholder. Plato's obsession with circles is legendary. His tone-circles are brilliant pedagogical constructions, but it is doubtful whether he invented any of the ones reconstructed here except possibly that for *Laws,* to be discussed in chapter 8. From the circle in figure 2 we can acquire our first insights into Platonic sexual metaphor and genetic theory. Modern algebraic notation for the arithmetic mean, $\frac{A+B}{2}$ and harmonic mean, $\frac{2AB}{A+B}$, is singularly unrevealing.

PLATO'S MUSICALIZED GENETIC THEORY

For Plato the "mother" principle is symbolized by the matrix of octave-doubles, hence also by the undivided circle. ("Universal nature molds form and type by the constant revolution of potency and its converse about the double.") Each revolution of the circle corresponds to a multiplication or division by 2. When Socrates asks, "And what about the many doubles? Do they look any less half than double?" he receives the obvious answer: "Each will always have something of both" *(Republic* 479b). In the tone-circle *multiples* and *sub-multiples* are "mixed up together" in order to "summon the intellect to the activity of investigation," an activity it undertakes only when "sensation doesn't reveal one thing any more than its opposite" *(Republic* 523b,c). Socrates, who narrates both questions and answers, is explicit:

> *Sight too saw big and little (or "the great and the small"), we say, not separated, however, but mixed up together. Isn't that so?*
>
> *Yes.*
>
> *In order to clear this up the intellect was compelled to see big and little, too, not mixed up together but distinguished, doing the opposite of what the sight did.*
>
> *(Republic* 524c,d)

By stipulating double meanings Plato achieves very great *mathematical* compression whose *musical* interpretation then functions as a "prelude to the song itself," the serious study of a *philosophy* founded on dialectics.

Plato is quite clear about what he means by "mother," "father," and "child" as mathematical metaphor:

> *We may fittingly compare the Recipient to a mother, the model to a father, and the nature that arises between them to their offspring.*
>
> (*Timaeus* 50d)

From his point of view "God" is the immovable "1," the reference point; "2" is "mother" or "Receptacle," symbolized by the undivided circle; and her first "child" is the arithmetic mean between them, the prime number 3 in the octave which has been doubled to avoid fractions so that it reads 2:3:4. But this child always has a "twin" brother derived from the reciprocal meanings of 3—functioning as harmonic mean—so that we actually meet our first Platonic children within the musical proportion 6:8::9:12. The numbers 8 and 9 representing harmonic and arithmetic means are "brothers" according to Socrates' definition:

> *Those born at the same time their mothers and fathers were procreating they will call sisters and brothers.*
>
> (*Republic* 461d)

Within any arithmetical double, which defines an octave, all integers linked by the generative ratios must be considered "brothers and sisters" because the alternate routes to them of multiplication (equivalent to alternate tuning strategies) make paternity quite uncertain except in this simple first case studied here. [In straightforward mathematical language, Plato is jesting about the functions of the principles of commutativity, $a \times b = b \times a$, and associativity, $a \times (b \times c)\ (a \times b) \times c$.] Marriages *must* always be to persons of "opposite temperaments" to preserve the framing intervals: note that $\frac{3}{2} \times \frac{4}{3} = 2$ (a union of the musical fifth 2:3 and fourth 3:4), and $\frac{5}{4} \times \frac{6}{5} = \frac{3}{2}$ (a union of the major third 4:5 and minor third 5:6 within the fifth 2:3). These "arranged marriages" are emphasized in the *Republic* (459-460), *Statesman* (304311), and *Laws* ("you should always prefer to marry somewhat beneath you," 773) as the most important responsibility of the legislators. (Note that in my examples, the first ratio is slightly larger than the second, and also that the numerator of the first is denominator of the second.)

In the metaphor of *Laws,* the prime number 3 "fathers" (generates) "citizens of the highest property class," defining musical fifths 2:3 and fourths 3:4, the largest subdivisions of the octave 1:2. The prime number

5 produces "citizens of the second highest property class" in the major thirds 4:5 and minor thirds 5:6 which subdivide the musical fifths 2:3. The prime number 7 fathers "citizens of the third highest property class" in the *septimal thirds* 6:7 and *septimal tones* 7:8, which subdivide the musical fourths 3:4. (Note that one *adds* intervals by multiplying their ratios expressed as fractions: $\frac{6}{7} \times \frac{7}{8} = \frac{3}{4}$.) Numbers generated by larger primes are excluded from ruling, and presumably constitute a "slave" class. Plato's jests about the sexual roles of numbers are endless, but underneath them lies a profound respect for the precision and elegance of Pythagorean musicalized number theory.

In political theory as in musical theory, both creation and the limitation of creation pose a central problem. Threatening infinity must be contained. Conflicting and irreconcilable systems, be they of suns and planets, of *even* octaves (powers of 2) and *odd* fifths (powers of 3), or of divergent political members of a *res publica*—must be coordinated as an alternative to chaos. What the demiourgos has shown to be possible in the heavens, what the musicians have shown to be possible with tones, the philosopher should learn to make possible in the life political. *Limitation,* preferably *self-limitation,* is one of Plato's foremost concerns. His four model cities correspond to four different tuning systems, each with its own set of generators and an explicit population limit:

City	**Callipolis**	**Athens**	**Atlantis**	**Magnesia**
Character	"celestial" (or "ideal")	"moderate" (or "best")	"luxurious" (or "worst")	"practicable" (or "second best")
Tuning	"tempered"	Pythagorean	Just	Archytas
Generators	2^p3^q	2^p3^q	$2^p3^q5^r$	$2^p3^q5^r7^s$
Limit	<1,000	≈ 20,000	12,960,000	5,040

PROCEDURE

The chapters which follow analyze Plato's mathematical allegories in detail and in the order which he carefully contrived for them. Some of the most puzzling elements in the marriage allegory, to be studied first, receive their explanation in the allegory of the thirty-seven guardians of Magnesia, to be studied last. The whole set of seven allegories is totally unified; not one of them can be understood except in the context of the others. I introduce whatever charts and diagrams seem useful in visualizing Plato's numerical operations, and I believe it will gradually become evident to the reader that Plato possessed and employed similar diagrams. Seldom does an interpretation depend upon the choice of translation, but where there is a difference of opinion concerning, the numbers in Plato's text I have consistently followed Robert Brumbaugh's emendations. The historical commentary which led me to these interpretations is generally relegated to Appendix I. Technical data which may aid the reader who wishes to make an independent study is summarized in Appendix II. The "marriage" numbers can be found in Appendix III, and a non-arithmetical introduction to the monochord in Appendix IV. A great difficulty arises from Plato's assumption that his readers (in his own time, his listeners) already knew the fundamentals of Pythagorean musical arithmetic. Few of my readers will possess such a background. I try to explain only what seems necessary at each step of the way, letting the full story come out only gradually as Plato carries us through a treatise on ratio theory which exhausts the musically interesting uses of the first ten integers. The facticity of number theory and the Pythagorean theory of the "means" leaves no room whatever for differences of opinion concerning correlations between tones and numbers. Such correlations are the straight-forward consequences of intitial assumptions. But Plato's metaphor demands imaginative interpretation, and his mythical commentaries on his own arithmetic speak to us only through our own intuition, hence we can never hope to achieve anything more certain than "the likely story." The story we pursue is Plato's prelude to philosophy, "the song itself." It is a story which the Platonic tradition has not known for two thousand years.

2
The Marriage Allegory
(REPUBLIC)

"Is it simply the case," asks Socrates, "that change in every regim comes from that part of it which holds the ruling offices—when faction arises in it?" Agreement from Glaucon, Plato's older brother, allows Socrates to proceed in Homer's style, illustrating "in what way . .. the auxiliaries and the rulers divide into factions against each other and among themselves." He prays the Muses for an explanation and then voices it for them, "with high tragic talk, as though they were speaking seriously, playing and jesting with us like children" *(Republic* 545d,e). Plato's language becomes deliberately archaic as he expounds on the implications of 60^4 = 12,960,000 as "sovereign of better and worse begettings" within the political framework of the *Republic.* This allegory, according to James Adam, is "notoriously the most difficult passage in Plato's writings."[1] Cicero labelled Socrates' sovereign number "numero Platonis obscurius" with more justice than he knew, for it is the arithmetical foundation for all later allegories, whose meanings are veiled by any obscurity of meaning here. The number itself is what Ernst Levy suggests calling a "tonal index." It is an arbitrary terminus for the potentially endless generation of tone-numbers, a *limitation* which provides integer expressions for some set of ratios. For a Pythagorean, it is a number which allows a particular set of fractions to be cleared within the octave 1:2 (i.e., in the way that "12," for instance, functioned as tonal index in allowing us to see arithmetic and harmonic means working together in the proportion 6:8::9:12).

The *Republic* is concerned with the meaning of justice in the ideal city, but when Socrates early in the dialogue describes that kind of city as "limited to essentials" to what is "necessary" Glaucon protests its lack of luxuries. To humor him, Socrates expands the discussion to include also the "luxurious, feverish city," later to be named Atlantis, doomed to destruction, however, because it is "gorged with a bulky mass of things which are not in the cities because of necessity" (363- 373) . The mathematical allegory we are about to study concerns not the "best" model Pythagorean tuning, limited to tones linked arithmetically by powers of 3 and musically by fourths and fifths—but also the "worst" model, Just tuning, idealized but impracticable because its pure thirds 4:5 and 5:6 are incommensurable with fifths 2:3 and fourths 3:4, giving rise to an internal conflict like that Socrates predicts between "rulers" and "auxiliaries." James Adam's analysis of Plato's arithmetic establishes the numbers which act like Levy's "tonal-indexes." Since the allegory occurs in Book VIII and follows very extensive discussions of arithmetic, music, dialectics, and political theory, Plato's own audience, probably already familiar with these musical examples, was also freshly oriented to the game he plans to play with them. Allan Bloom's translation of the text is broken into sections a), b), c)...to facilitate analysis.

THE ALLEGORY

a) Since for everything that has come into being there is decay, not even a composition such as this will remain for all time; it will be dissolved. And this will be its dissolution:

b) bearing and barrenness of soul and bodies come not only to plants in the earth but to animals on the earth when revolutions complete for each the bearing round of circles; for ones with short lives, the journey is short; for those whose lives are the opposite, the journey is the opposite. Although they are wise, the men you educated as leaders of the city will nonetheless fail to hit on the prosperous birth and barrenness of your kind with calculation aided by sensation, but it will pass them by, and they will at some time beget children when they should not.

c) For a divine birth there is a period comprehended by a perfect number;

d) for a human birth, by the first number in which root and square increases, comprising three distances and four limits, of elements that make like and unlike, and that wax and wane, render everything conversable and rational.

e) Of these elements, the root four-three mated with the five, thrice increased, produces two harmonies.

f) One of them is equal an equal number of times, taken one hundred times over. The other is of equal length in one way but is an oblong; on one side, of one hundred rational diameters of the five, lacking one for each; or, if of irrational diameters, lacking two for each; on the other side, of one hundred cubes of the three.

g) *This whole geometrical number is sovereign of better and worse begettings. And when your guardians from ignorance of them cause grooms to live with brides out of season, the children will have neither good natures nor good luck. Their predecessors will choose the best of these children; but, nevertheless, since they are unworthy, when they, in turn, come to the powers of their fathers, they will as guardians first begin to neglect us by having less consideration than is required, first, for music, and, second, for gymnastic; and from there your young will become more unmusical.*

(Republic 546a-d)

INTERPRETATION

a) The warning that degeneration is inevitable. Plato's theory that even the best aristocracy will degenerate in time through a timocracy, oligarchy, and democracy into a tyranny colors much of the *Republic.* Any tuning which uses the "perfect" ratios of integers—and Socrates' system uses 1:2:3:4:5:6 will degenerate unless the number of tones is rigidly limited. A series of perfect fifths 2:3, for instance, slightly larger than $\frac{7}{12}$ of the octave, could agree with the octave series only if some higher power of 2 agreed with some higher power of 3, an obvious impossibility since the first series is *even* and the second is *odd.* Pure thirds 4:5, slightly smaller than $\frac{4}{12}$ of the octave, can never quite agree either with octaves or fifths for the simple arithmetical reason—which the tuner experiences aurally that $2^p = 3^q = 5^r$ only at the zero power, equated as 1 and symbolizing God by its own absolute invariance. The political lesson—a musical analogue—points to the impossibility of founding a lasting state on any model which lacks an internal principle of "self-limitation," an element deliberately omitted from Socrates' formula.

b) Cycles of bearing and barrenness. Since "universal nature molds form and type by the constant revolution of potency and its converse about the double" *(Epinomis* 990e), it is easy to recognize that powers of 2 coincide in the circle and thus produce, by themselves, only cycles of barrenness, i.e., empty octaves, a "pre-musical" condition. Cycles of bearing will require the participation of other primes to "father" our tone children. "Calculation aided by sensation," however, will not prevent the birth of unmusical children in the arithmetic which follows; for whether we tune by ear or by calculation, we shall meet discrepancies arising from Socrates' withholding any principle of limitation while using a formula with an internal contradiction; his ratio 4:3 is incommensurable with 4:5.

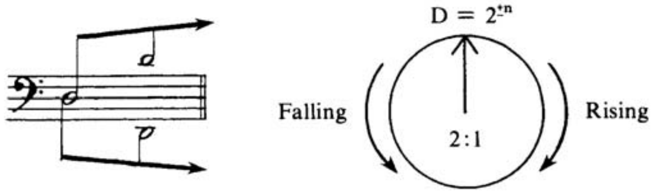


FIGURE 3

Cycles of Barrenness
Multiplication and division by 2 can only produce octaves of the reference tone.

c) The perfect number comprehending divine births. The perfect number to which Plato alludes, I believe, is the very first one, 6 (= 3 + 2 + 1, i.e., as the *sum* of its proper divisors).[2] The ratios of the first six integers-1:2:3:4:5:6—can define all the tones of the Greek Dorian mode, Plato's "true Hellenic mode" *(Laches* 188d), musical foundation for the *Republic* (398-399), and its *reciprocal,* our modern major mode.

Greek Dorian	D	c	b♭	A	G	f	e♭	D	(falling)
Reciprocal Dorian	D	e	f♯	G	A	b	c♯	D	(rising)
Ratios:	1			:				2	
	2		:		3	:		4	
	4	:	5	:	6				
					4	:	5		
		5	:	6					
				4	:	5	:	6	

In the falling scale these numbers function as ratios of string length. In the rising scale they function as ratios of frequency, or, from the Greek point of view, as divisors of string length (in "sub-contrary" or "harmonic" progressions). This tuning has come down through history as Ptolemy's "diatonic syntonon."[3] Socrates' formula will lead us to it via an indirection which teaches us several lessons along the way. Unless we are already acoustical theorists, only in retrospect will we appreciate why this model for Just tuning functions perfectly (i.e., it produces no "unmusical children") within the limit of Socrates' perfect number 6. That these particular births are "divine" we can understand later from *Critias* where they are "fathered" directly by Poseidon, if my story is "the likely one."

d) The three increases which make human births conversable and rational. "Root and square increases" suggest *geometrical* progression through the powers of whatever number is involved—$x:x^2:x^3:x^4$—to three distances and four limits," so that we learn whatever lesson Socrates intends to teach us at the "solid dimension," in accordance with Plato's theory of perception:

> *But the condition under which coming-to-be universally takes place what is it*
>
> *Manifestly 'tis effected whenever its starting point has received increment and so come to its second stage, and from this to the next, and so by three steps acquired perceptibility to percipients.*
>
> (Laws 894)

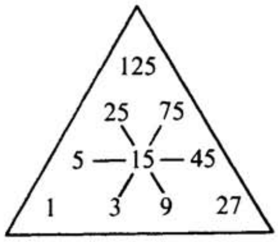
The point-line-plane-solid progression is symbolized by the Pythagorean tetractys, which Nicomachus teaches us to interpret as interlocked geometrical progressions:

point	∘	(1)
line	∘ ∘	A B
plane	∘ ∘ ∘	A^2 AB B^2
solid	∘ ∘ ∘ ∘	A^3 A^2B AB^2 B^3

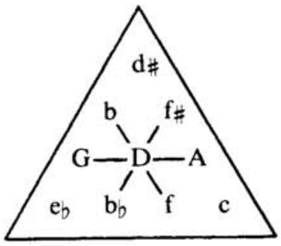
Figure 4 consists of Nicomachean arrays relevant to Socrates' arithmetic in this allegory; they are alternate matrices for Just tuning *if we assume octave equivalence,* a standard assumption in tuning theory. (Please note that figures 4a and 4c, which show radical numerical differences, are equivalent only when we assume that powers of 2 are tonally irrelevant.) All of Socrates numbers can be studied within such planimetric arrays, and all participate interchangeably as *arithmetic, harmonic,* and *geometric means* in the alternate perspectives on musical reality to which Plato is leading us.

Numbers produce "like and unlike" tones by their reciprocal functions as multiples and submultiples (of either string length or frequency). Numbers "wax and wane" in falling-rising tone progressions according to whether we order them in increasing or decreasing size, and conceive of them as multiples or submultiples of string length. All tones are defined by *rational* numbers which become "conversable" when based on a common denominator.

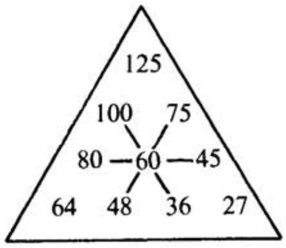
e) The formula: 4:3 mated with the 5, thrice increased. If we reason like the Platonists that "4:3 mated with 5" means $3 \times 4 \times 5 = 60$, and that three "root and square increases" means— $\times\ 60 \times 60 \times 60 = 12{,}960{,}000$, i.e., 60



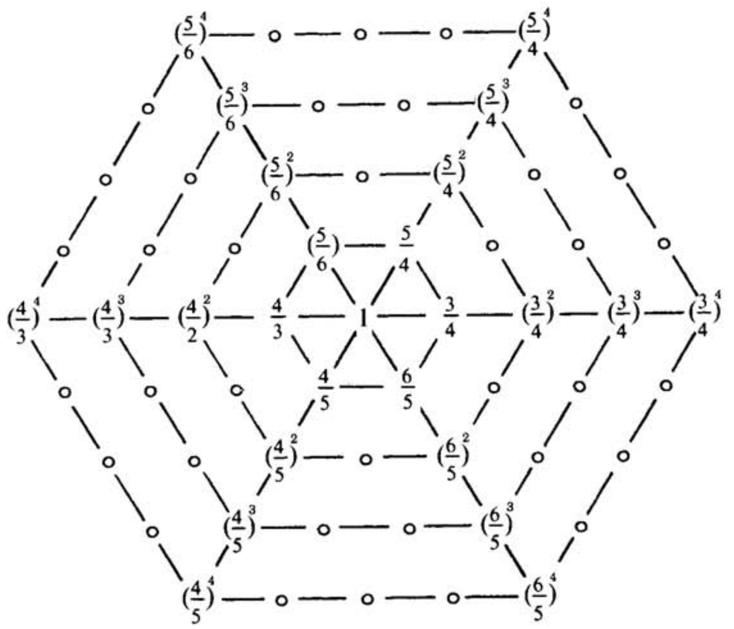
a) Smallest integers
$3^p \times 5^q$



b) Equivalent tones



c) 4:3 mated with 5



d) Hexagonal symmetry

"thrice increased" to give "three distances and four limits," we should arrive safely at Socrates' sovereign number but would miss all the musical lessons along the way, lessons he underlines by the curious factoring he does next. Only by proceeding with great circumspection here can we understand what follows. Since we have already discovered, in figure 2, that the reciprocal tonal meanings of the "root 4:3" become "conversable" within the musical proportion 6:8::9:12 (equivalent to using 12 as *least common denominator* for $\frac{4}{3}$ and $\frac{3}{4}$), the new question is this: How can "5" be mated with the musical proportion?

That the "human male" prime number 5 enters harmonic theory as *arithmetic mean* within the perfect fifth of ratio 2:3—expanded to 4:5:6 to avoid fractions—Socrates has just warned his audience a few pages earlier. When he declares that prospective "guardians" must prepare for their civic duties by spending on the rigorous study of dialectics no less than "double the number of years devoted to gymnastic," he is asked, "Do you mean six years, or four?" (6:4 = 3:2) "Don't worry about that," he replies. "Set it down at five" (539d,e). Our musical problem, then, is to discover in how many ways the ratios 4:5 and 5:6 can be fitted into the ratio 3:4 which occurs twice in the musical proportion 6:8::9:12. These two new values create exactly four patterns, as shown in figure 5, according to whether the new ratios are taken *after* the first tone or *before* the second, inseparable from the question as to whether the pitch sequence rises or falls.

To musicians the ratio 4:5 defines a "major third," the ratio 5:6 defines a "minor third," and the octave sequences in figure 5 are various "pentatonic" (5-tone) scales. Notice how Socrates' "marriage" patterns "wax and wane"; pitch rises or falls according to whether the numbers increase or decrease within their respective "doubles," and how they are interpreted on the string. Each number set is "friends" with another in the sense that they "share all *property (*= tones) in common" *(Laws* 739c). These two pairs of "friends" are harmonic or "sub-contrary relatives" of each other:

FIGURE 4

Equivalent Matrices for Just Tuning

Smallest integers in matrix a) are in "continued geometric progression" via powers of 3 and 5; so also are the integers in matrix c) via powers of $\frac{4}{3}$ and $\frac{5}{4}$, these being only two out of many possible arrangements of the equivalent tones b). The hexagonal matrix d) suggests how the prescribed ratios develop perfect inverse symmetry around the reference tone D as far as "three distances and four limits" from the initial symmetric pattern. Musicians will find it easier to think in terms of fourths and fifths ↔, major thirds ⤢, and minor thirds ↘.

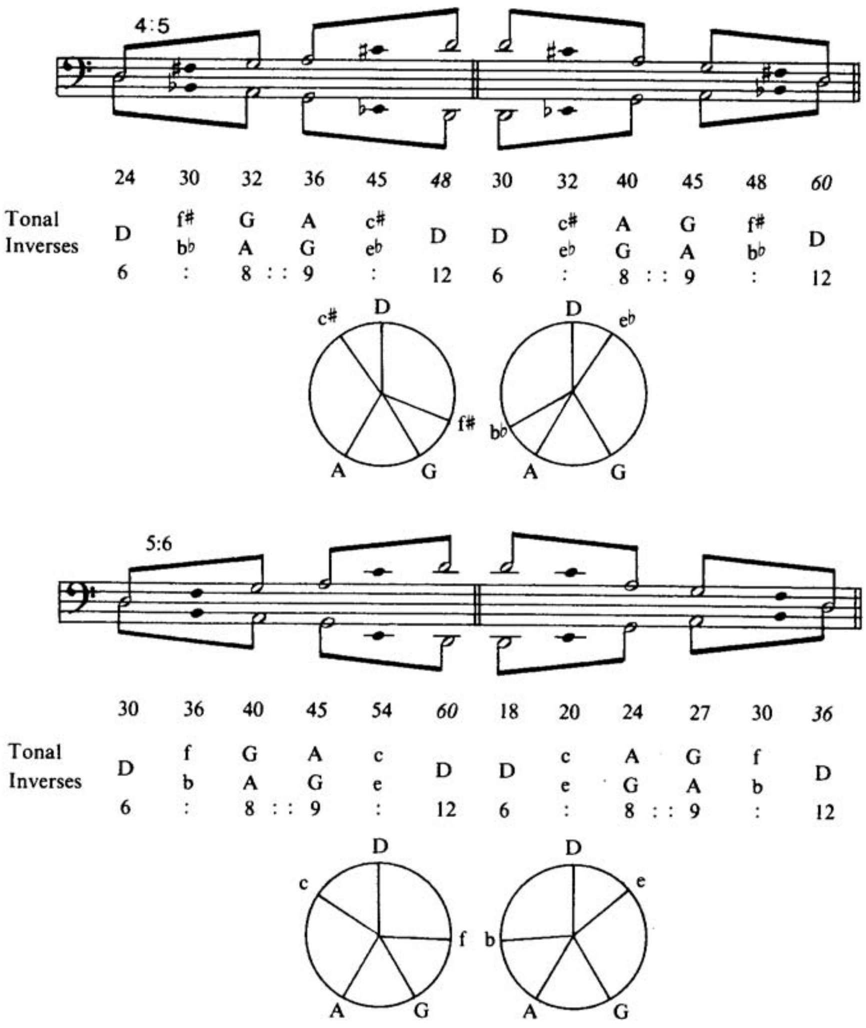


FIGURE 5

A Platonic Wedding

Numbers which define one pentatonic sequence are "friends" with the reciprocals of a second set, and the pair of number sets is related by the same ratios (4:3 mated with 5) to still another pair of similarly related sequences. These four patterns exhaust the musically useful marriages available under Plato's rules.

"the root 4:3"	6	:	8	::	9	:	12					
	12	:	9	::	8	:	6					
"friends"	24	:	30	:	32	:	36	:	45	:	48	"waxing"
4:5	60	:	48	:	45	:	40	:	32	:	30	"waning"
"friends"	30	:	36	:	40	:	45	:	54	:	60	"waxing"
5:6	36	:	30	:	27	:	24	:	20	:	18	"waning"

Plato legislates *pentatonic* "wedding feasts" in a way which fully justifies the nickname "marriage allegory" attached to this fragment of the *Republic:*

> *Neither family should invite more than five friends of both sexes, and the number of relatives and kinsmen from either side should be limited similarly.*
>
> (*Laws 775*a)

The four pentatonic marriage patterns contain a total of eleven different tones belonging to the Greek Dorian scale and its reciprocal and to a set of numbers which, for the first time, makes clear to us why the "guardians" of a musical city begin training in dialectics at age 30 and complete their preparation as future "rulers" in their 50's (539-540).

"Guardians"	30	32	36	40	45	48	54	60
Greek Dorian	D	e♭	f	G	A	b♭	c	D
Reciprocal Dorian	D	c♯	b	A	G	f♯	e	D

The eleven different tones in these reciprocal diatonic scales can be coalesced into one chromatic scale whose limiting octave-double (Levy's "tonal index") is factorial six (6!) = 720.

Tones:	D	e♭	e	f	f♯	G	A	b♭	b	c	c♯	D
"Waxing"	360	384	400	432	450	480	540	576	600	648	675	720
"Waning"	720	675	648	600	576	540	480	450	432	400	384	360

At this level of development, the scale "bent round into a circle" as required in *Timaeus* (36c), can function as a zodiacal symbol (Fig. 6). Since these tones are distributed within the octave in perfect inverse symmetry, the same set of integers serves both the rising and falling scales: they could be applied to the tone circle in either direction. The 360 arithmetical subdivisions within the octave 1:2 = 360:720 correlate with the 360 days in the "schematic" years of various ancient peoples.

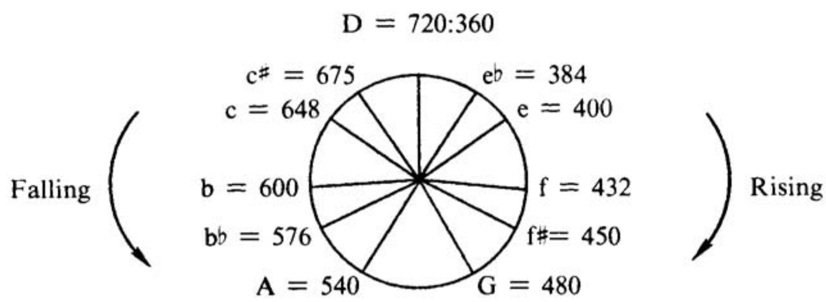


FIGURE 6

The Chromatic Scale as a Tonal Calendar

Ptolemy, whose own tonal zodiac is the oldest we have inherited, assigned 36,000 years to the precession of the equinoxes, much too large a number, calling it a Platonic "great year."[4] Notice that 360 × 36,000 = 12,960,000, Socrates' "sovereign number.[5]

We are watching Taylor's "perfect symmetry" develop within Brumbaugh's "modulus" of the octave, tones being generated by the ratios within Socrates' "perfect number" 6, while fractions are cleared by a succession of Levy's "tonal indexes." We shall meet these tones again in *Critias* where they symbolize, I believe, "Poseidon and his five pairs of twin sons," and then once more in *Laws* where they are "boundary markers" for Magnesia as well as "guardians from the parent city." No other Greek tuning possesses so many delightful numerical correspondences with calendrical periods and the lives of men.

As we multiply 60 by itself to gain the larger tonal indices of 60^2, 60^3, and finally 60^4 = 12,960,000, the integers available to a mathematical harmonics based on the ratios of the first six integers increase dramatically. Each multiplication provides new ratios of 3:4, 4:5, and 5:6 *in each direction* from the tones and numbers we already have, equivalent to providing for musical *transpositions* both higher and lower, respectively, by perfect fourths, major thirds, and minor thirds. These tonal transpositions appear as arithmetical *translations* and can be graphed as geometrical *rotations* of the numbers and patterns of figures 5 and 6. (The wholetone ratio 8:9 is very slightly larger than $\frac{1}{6}$th of the circumference of our circle, as the next allegory will demonstrate, and would have provided Socrates an easy clue to the positions of other ratios had he actually desired to draw the tone-circles introduced here to make clear his meaning.) Before studying the tonal consummation of 60^4, let Socrates first confirm that the tonal indices of 36, 48 and 60 (see fig. 5) are actually the ones he has in mind.

f) The two harmonies. A harmony, literally a "joining together," is the product of multiplication. Our tonal indexes appear in James Adam's analysis:

(a) $12,960,000 = 3600 \times 3600 = (36 \times 100) \times (36 \times 100)$

(b) $12,960,000 = 4800 \times 2\,700 = (48 \times 100) \times (27 \times 100)$

In the first harmony (equal an equal number of times, taken one hundred times over) we see $(36 \times 100)^2$. In calling attention to the factor of 100 Plato also isolates the factor of 36, the numbers 36 and 60 being indexes for the second pair of scales in figure 5. The second harmony, 4800 × 2700 is described as "oblong." The factor of 2700 is "one hundred cubes of the three" (that is, $3^3 = 27$, and 100 × 27 = 2700), a factoring which isolates 27 as the Socratic tonal index for children of the very "best" births, as we shall learn in chapter five from *Timaeus,* where the cube of 3 generates all seven tones:

The cube of 3	1	3	9	27
"Waxing as integers"	D	A	E	B
"Waning as reciprocals" D		G	C	F

In scale order, these tones constitute the Greek Phrygian mode, the only mode beside the Dorian which Plato admits into the ideal city of the *Republic.* Its rising and falling patterns of tones (t) and semitones (s) are identical:

rising	D		E		F		G		A		B		C		D
		t		s		t		t		t		s		t	
falling	D		C		B		A		G		F		E		D

How this tuning emerges within the "sovereign" number $60^4 = 12,960,000$ we shall see in a moment. We have yet to examine the curious description of 4800.

The factor of 100, occurring again in 4800, makes that number "of equal length in one way" to the others. The factor of 48 is the "rational diameter of the five, lacking one" in the sense that the rational approximation to the *diagonal* (Socrates' "diameter") *of a square* whose sides are five units is the *square root* of 49, which "lacking one" yields 48. The "irrational diameter" is the square root of 50, which "lacking two" yields 48, the third important tonal index of Figure 4. Adam's diagram clarifies the language (Fig. 7). Plato's

FIGURE 7

Diameters of the Square

elaborate allusions to the numbers 48, 49, and 50 will be remembered when we study the guardians in *Laws* (in chapter eight) and discover that his "citizens of the third property class" (generated by the prime number 7) actually enclose the square root of 2 within these numbers treated as "tone-numbers." Here the essential tonal indexes are 27, 36, 48, and 60; the extraneous factor of 100 which occurs repeatedly will reappear in chapter seven as the "playful" factor in Atlantis, magnifying everything in that city to stupendous size. Now we must look for Socrates' genetic lesson.

g) The conclusion that the number under discussion is "sovereign." "This whole geometrical number," Socrates concludes, "is sovereign of better and worse begettings." We must look, then, for the *limit* of both "better and worse" musical births, not forgetting that Socrates' predicted inevitable dissension even among his "ruling classes." To help the reader contemplate the possible tonal meanings of numbers $2^p3^q5^r \leq 12{,}960{,}000$, I introduce here a fragment of the multiplication table for 3×5, arbitrarily cut off at that limit, using the symbol ° for counters, the symbol + for reciprocals, and capital letters for those tones "fathered" by the divine male number 3 which generates "rulers" or "citizens of the highest property class," meaning fifths 2:3 and fourths 3:4, and small letters for tones fathered by the human male number 5 which generates "auxiliaries" or "citizens of the second highest property class," meaning major thirds 4:5 and minor thirds 5:6. We can safely eliminate all factors of the female number 2 because they are merely octave replications of tones generated here by the odd, *male* numbers. The solid lines in figure 8 enclose products within Plato's successive limits of 60, 60^2, 60^3 and 60^4. The broken lines show the range of reciprocals within the sovereign limits, 89 products actually defining 121 different pitches. "Rulers" appear along the central axis ; guardians appear in neighboring rows. The numbers can be computed by taking the counter in the lower left-hand corner of each table as the unit 1, with powers of 3 graphed along the base →, powers of 5 along the diagonal ↗, and powers of 2

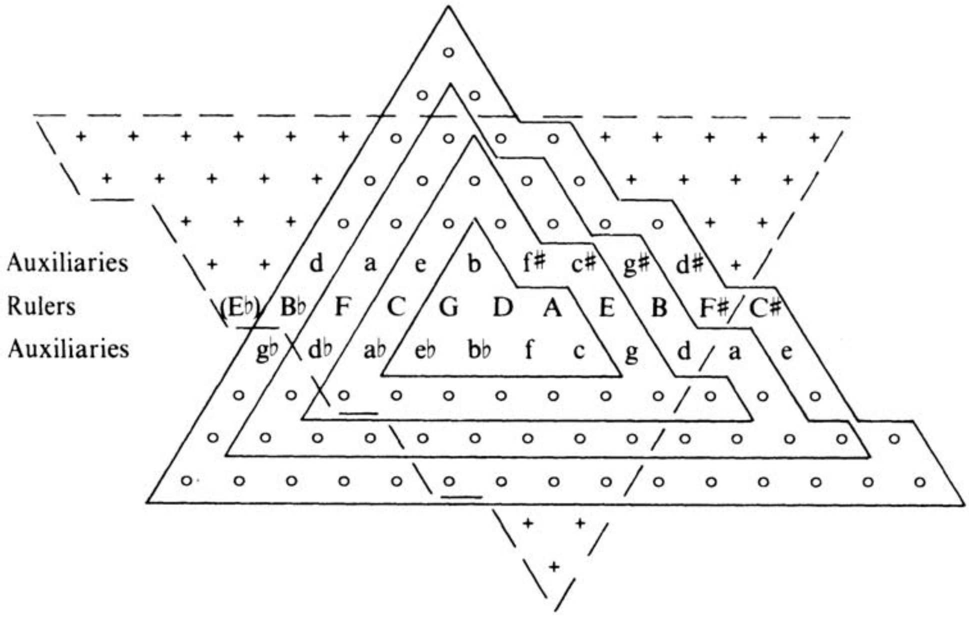


FIGURE 8

Numbers 3^p5^q < 12,960,000 in Logarithmic Arrays

(This is the multiplication table for 3 × 5. The counter in the lower left corner of each array symbolizes 1; powers of 3 extend to the right and powers of 5 extend upward along the diagonal, as in figure 4a. All of these numbers play reciprocal roles around the geometric mean = 1 = D in the center, transformed into successive powers of 60 in Plato's context Equidistant elements in any direction are in geometric progression. Rulers in the central axis top at E♭ on the left and C'♯ on the right. Auxiliaries in adjacent rows differ from rulers laving the same tone names by the ratio of the syntonic comma 80:81, a micro-interval symbolizing the dissension which arises within the ruling classes when the original eleven-one limit is exceeded. Every triangular array of ten elements possess the alternate implications of the Nicomachean matrices in figure 4, and the table as a whole is now large enough to permit reciprocal matrices to be developed from every one of the original eleven tones.

imagined as projecting these numbers into any arithmetical double whose "numerosity" is large enough to contain them. In the center of the graph is Socrates' "seat in the mean," *(geometrical* mean), four rows up from the base and four places to the right, hence containing factors of both 5^4 and 3^4 (i.e. as well as 4^4 which does not need to be represented here). The only row whose tonal meanings are invariant under reciprocation is the central axis itself, limited to Pythagorean tuning. The Just "auxiliaries" above and below exchange places under reciprocation.

The ten numbers along the horizontal axis of Figure 8 are "rulers" in the *Republic* and become the "army of Ancient Athens" in *Critias,* to be studied in their separate numerical context in chapter 6. They generate the same eleven tones we met in figures 5 and 6—forming the Dorian scale and its reciprocal but in a slightly "improved" Pythagorean tuning, somewhat closer to equal temperament than the Just tuning we met earlier, and thus making a more equitable distribution of space in Plato's circular cities. This series of "best" births stops at E♭ in one direction and C♯ in the other, just short of the internal disagreement which would arise between A♭ and G♯ in the next expansion, a problem Socrates dramatizes for us in the following Tyrant's allegory. (The Phrygian tones F C G D A E B are a symmetric sub-set.)

But look at the dissension which has arisen between rulers and auxiliaries. Tones sharing the same letter names in adjacent rows (and differentiated here by the use of capital and small letters) actually differ in pitch by the ratio of the syntonic comma 80:81, a micro-interval too small to differentiate *different* tones for the ear, yet too large for us to maintain the illusion that they have exactly the *same* pitch. (Computation: from D to f♯ along the diagonal, ↗ , is a pure third of ratio 4:5; from D to F♯ along the horizontal axis, →, is a *ditone* third of $(9/8)^2 = 81/64$. Since $5/4 = 80/64$, the comma has the value 80:81. The difference is 22 *cents,* about one-fifth of a semitone.) Similar commas exist throughout the table between pairs of elements having the same planimetric spacing, and such commas accumulate from one row to the next.[6] Here in plain alphabetical view is the "dissension" foretold among the ruling classes. A glance at the extent of the table is enough to suggest that it defines numerically far more pitches than any musician could recognize in a useful way. Socrates' formula is generating "unlikeness and inharmonious irregularity" (547a). Our "young," as he claimed, have "become more unmusical." Three generations via $5/4$ (i.e., three places along the diagonals ///, both above and below the "founders") produces "great grandchildren" at the *diesis* 128:125, almost a quartertone short of the octave: $(5/4)^3$=125:64, instead of the octave 128:64. In Plato's metaphor, the highest and lowest rows of the table are filled with mad relatives" and "children of worse births" who must be excluded from rule in a musical city.

But Socrates does not propose to leave us in this dilemma. He rescues us from this musical distress in the Tyrant's allegory which follows almost immediately. In *Critias* (my chapters 6 and 7), Plato will resume this story with an extended commentary on its arithmetic under the mythical veil of Atlantis. And in *Laws* (chapter 8), he will show us not only new uses for our "wedding party" and new meanings for the numbers 48, 49 and 50, but also new reasons for considering 60^4 = 12,960,000 "sovereign" in his kind of musical politics. The marriage allegory will prove to dominate our whole adventure in imagination, finally providing us, in chapter 9, with a complete set of the Pythagorean triangles ancient geometers once required, and a surprising link to the most famous mathematical cuneiform tablet of ancient Babylon.

3
The Tyrant's Allegory
(REPUBLIC)

"Surely some terrible, savage, and lawless form of desires is in every man, even in some of us who seem to be ever so measured," Socrates remarks, deflecting the argument from dissension within the city to dissension within the soul *(Republic* 572b). When he then calculates the measure of a self-indulgent tyrant's "suffering" as exactly 729 times that of a philosopher, "king of himself," he is playing a game strictly in accordance with the rules we have noted. Plato's dialectics of opposites require that such a number be taken in both directions from a reference "1" (i.e. to observe both "potency and its converse"), then made conversable and rational" by a least common denominator, and finally interpreted *within* the octave 1:2, his "cycle of bearing." This allegory proves to be the simplest of all of Plato's mathematical *allegories. He* leads us directly to the ratio *531441:524288 known historically as the* "Pythagorean comma," a flaw even his "rulers" in the marriage allegory would have suffered among themselves if the generation of tone-numbers along the central axis of figure 8 had not been abruptly halted where it was, that is, before A♭ appeared on the left and G♯ on the right, programmed for the very next expansion. This comma is the internal dissension the Tyrant suffers because he violates the principle of self-limitation.

If Plato's arithmetic proves easy and his lesson obvious, the tyrant's allegory still rewards meticulous study by revealing the great care Plato took to be understood—notwithstanding his jests—and by confirming in new language the principles we have already discovered.

In his preamble to this allegory, Socrates names five kinds of rulers—kingly, timocratic, oligarchic, democratic, and tyrannic—and demands that Glaucon declare who, in his opinion, "is first in happiness, and who second, and the others in order, five in all." Glaucon replies:

> *The choice is easy. For with respect to virtue and vice, and happiness and its opposite, I choose them, like choruses, in the very order in which they came on stage (580b).*

This naive answer inspires Socrates to lead Glaucon through an extended lesson on the differences between reality and appearances, stressing particularly the purely relational meanings of words like "up," "down," and "middle." Only when Glaucon's initial confidence in appearances is thoroughly dissipated, Socrates springs on him his "hedonistic calculus"[1]: Do you know how much more unpleasant the tyrant's life is than the king's?" "I will, if you tell me," Glaucon answers. Here is Allan Bloom's text of the allegory, sectioned for analysis:

THE ALLEGORY

(a) "There are, as it seems, three pleasures—one genuine, and two bastard. The tyrant, going out beyond the bastard ones, once he has fled law and argument, dwells with a bodyguard of certain slave pleasures; and the extent of his inferiority isn't at all easy to tell, except perhaps as follows."

"How?" he said.

(b) "The tyrant, of course, stood third from the oligarchic man; the man of the people was between them."

"Yes."

"Then wouldn't he dwell with a phantom of pleasure that with respect to truth is third from that other, if what went before is true?'"

"That's so."

"And the oligarchic man is in his turn third from the kingly mail. if we count the aristocratic and the kingly man as the same."

"Yes, he is third."

"Therefore," I said, "a tyrant is removed from true pleasure by a number that is three times three."

"It looks like it."

(c) "Therefore," I said, "the phantom of tyrannic pleasure would, on the basis of the number of its length, be a plane?"

"Entirely so."

"But then it becomes clear how great the distance of separation is on the basis of the square and the cube."

"It's clear," he said, "to the man skilled in calculation."

(d) *"Then if one turns it around and says how far the king is removed from the tyrant in truth of pleasure, he will find at the end of the multiplication that he lives 729 times more pleasantly, while the tyrant lives more disagreeably by the same distance."*

"You've poured forth," he said, "a prodigious calculation of the difference between the two men the just and the unjust in pleasure and pain."

(e) *"And yet the number is true," I said, "and appropriate to lives too, if days and nights and months and years are appropriate to them."*

"But, of course, they are appropriate," he said.

"Then if the good and just man's victory in pleasure over the bad and unjust man is so great, won't his victory in grace, beauty, and virtue of life be greater to a prodigious degree?"

"To a prodigious degree, indeed, by Zeus," he said.

(*Republic* 587b-588a)

INTERPRETATION

(a) Three pleasures one genuine, and two bastard. Plato's one realm of genuine pleasure is the model octave 1:2, his matrix cycle of "bearing and barrenness." His two "bastard pleasures" are the larger doubles 2:4 and 4:8 which occur *in passing* in some formulas (as in the point-line-plane-solid progressions 1:2:4:8 of *Timaeus* and *Epinomis).* The tyrant, to whom Socrates now assigns the number 9, lies beyond these bastard pleasures. We are being bluntly warned, then, that the numbers which occur in this allegory must be subjected to "octave reduction," that is, they must be projected into one circle and given integer values within one "double."

(b) The tyrant is removed from true pleasure by a number that is three times three. By a verbal trick, Socrates renumbers his five kinds of rulers, knowing full well that his audience, prepared by the preceding discussions of music and mathematics and the elaborate marriage allegory will read a tonal meaning into every number. By changing Glaucon's manner of counting, Socrates arrives at the number 9.

Rulers	King	Timocrat	Oligarch	Democrat	Tyrant
Glaucon	1	2	3	4	5
Socrates	1	2	3 = 1	(6) 2	(9) 3

This seeming nonsense alerts a Pythagorean to the fact that Socrates has suddenly eliminated the "human male number 5"—which caused all the dissension in the marriage allegory from his present formula for the soul.[2] Lest anyone misunderstand, he simply tells us that the formula begins with a number "that is three times three" (= 9). The "waxing-waning" implications of the formula can be understood tonally by translating the numbers and their reciprocals directly into tones:

	1	2	3	6	9
rising	D	D	A	A	E
falling	D	D	G	G	C

Bastard and slavish "pleasures" can be eliminated by projecting the tones into the tone circle where redundant doublings coincide, "potency and its converse" being graphed as rotation and counter-rotation (Fig. 9).

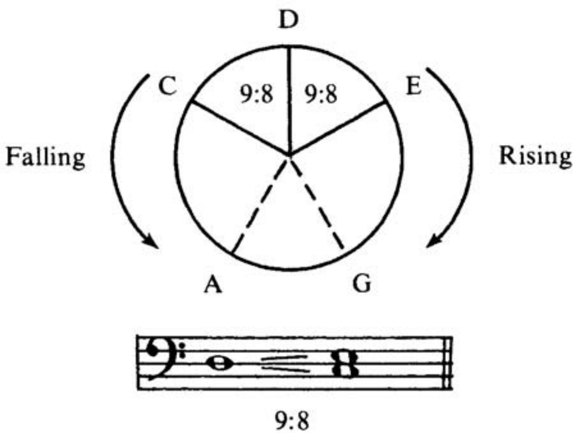


FIGURE 9
The Elimination of "Bastard" and "Slavish" Pleasures

(c) Nine as a plane number, reinterpreted as a line number. To Socrates, the number 9 is a plane number because he has reached it by multiplying two numbers. Now he asks us to reinterpret 9 as a "line" number in the point-line-plane-solid progression through the square and cube, so that we arrive at his lesson at 9^3, the "solid" dimension. Since 9 actually reduces to a wholetone of 9:8, its cube will reduce to $(9/8)^3 = 729/512$, a Pythagorean approximation to the square root of 2, a problem which fascinated Socrates in the marriage allegory. Here are the powers of 9 together with their reciprocal

tonal meanings and their numerical reduction:

Generative formula:	$9^0 = 1$	9	81	729
rising tones	D	E	F♯	G♯
falling tones	D	C	B♭	A♭
smallest integers	512	576	648	729
analysis	(2^9)	$(2^6 9)$	$(2^3 9^2)$	(9^3)

In the tone circle below we see that our reciprocal "tritone" progressions (three wholetones of 8:9 in both directions from D) overlap by a small interval known as the Pythagorean comma. This is the total "excess and deficiency" (G♯ :A♭) between the true square root of 2 (G: = A♭ in equal temperament) and Socrates' approximations.

d) A prodigious calculation. So far we have actually done no important calculation at all; Socrates has always supplied the numbers. What remains to be done is to measure the size of his comma, and the answer—in six-digit numbers—must have seemed a prodigious calculation at some point in the history of mathematics. Our reciprocal triton progressions must be turned into one linear progression of six consecutive wholetones, a task actually done for us in proposition IX of th *Sectio Canonis,* now attributed to Plato's friend Archytas. The smallest integers for six consecutive wholetones of 8:9 will lie between 8^6 and 9^6:

$8^6 = 262144 = A\flat = 512^2 = 2^{18}$; $\times 2 = 524288$

$\times\ 9/8 = 294912 = B\flat$

$\times\ 9/8 = 331776 = C$

$\times\ 9/8 = 373248 = D$ (= *center of symmetry*)

$\times\ 9/8 = 419904 = E$

$\times\ 9/8 = 472392 = F\sharp$

$\times\ 9/8 = 531441 = G\sharp = 729^2 = 9^6$

By octave reduction we see that the Pythagorean comma is worth exactly 524288:531441, approximately 73:74. This is the amount by which six wholetones of 8:9 exceed the octave 1:2, introducing discord into our realm of perfect pleasure because the two pitches are far enough apart to destroy any illusion that they are the same, yet too close together for the ear to differentiate them in practice. (This comma is 24 cents; the syntonic comma was 22 cents.)

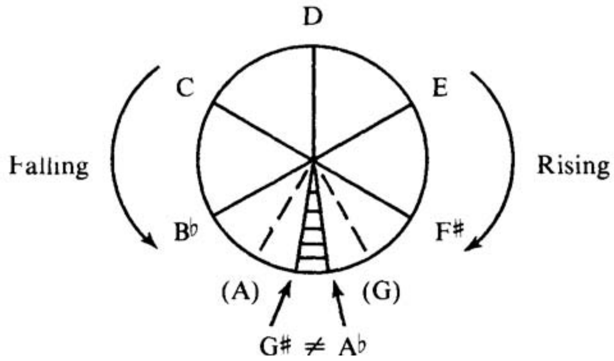


FIGURE 10

The Cube of Nine as $\sqrt{2}$

Notice Socrates' language: the king lives 729 times more pleasantly and the tyrant 729 times more disagreeably. The number 729^2 is actually the measure of G♯ from A♭ $= 1 = 8^6 = 512^2 = 524288 = 2^{19}$, etc. , all numbers 2^p being cyclic identities in Platonic tone circles.

e) A true number appropriate to lives. The comma we have defined exceeds the octave 1:2 by about the same ratio by which Plato's year of 365 days exceeds his calendar base of 360 (12 months of 30 days each), a problem he solves in *Laws* in the Egyptian manner by adding five extra “election” days at the end. The comma of 73:74 and the ratio 360:365 = 72:73 show the same discrepancy in the soul as in “days and nights and months and years.” It is amusing that Philolaus, the teacher of Plato's Pythagorean friend Archytas, counted 364½ days in a year so that his days and nights together totalled 729, a number essential to Pythagorean musicology, however badly it fitted the facts of calendar making.

But what conclusion are we supposed to draw from all this? The kingly man will avoid this problem in his soul, Socrates claims, by declining to “give boundless increase to the bulk of his property and thus possess boundless evils.” He will look fixedly at the regime within him and guard “against upsetting anything in it by the possession of too much or too little substance” (Bloom) or “excess or deficiency of wealth” (Shorey) *(Republic* 591c-e). It is the last increase which leads to musical disaster, to the tritone “diabolus in musica” (“devil in music” the worst possible dissonance). It lies outside the limits of the Greek tetrachord frames of 3:4, and *within* the “disjunction” 8:9 of the musical proportion 6:8::9:12. If we restrain ourselves, like the kingly man (or “the

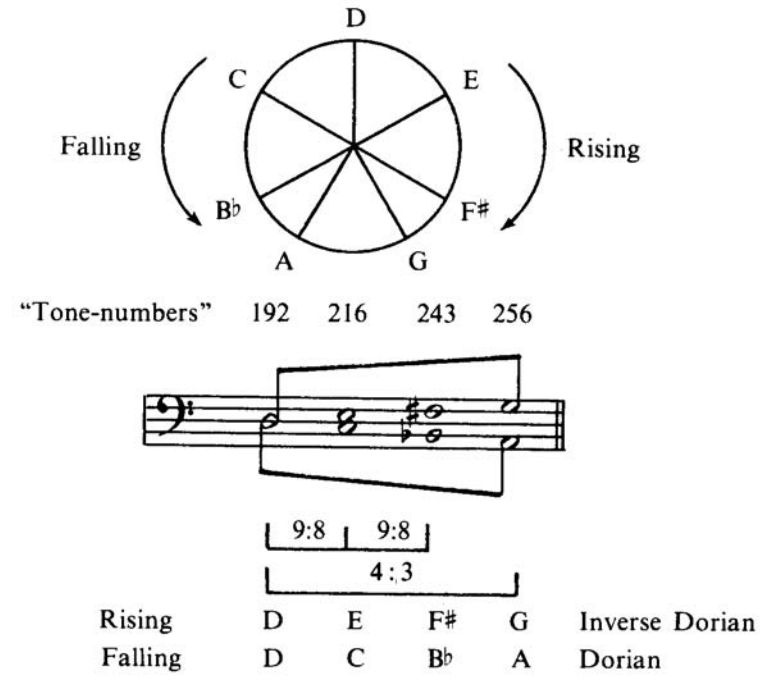


FIGURE 11

The Dorian Tetrachord and its Reciprocal

philosopher king of himself") we have a model Dorian "tetrachord" (four strings) which Plato will later introduce into *Timaeus* as the model for the "World-Soul."

Glaucon doubts that the model for such personal or civic "self-restraint" which Socrates preaches "exists anywhere on earth." "But in heaven," Socrates rejoins, "perhaps, a pattern is laid up for the man who wants to see and found a city within himself on the basis of what he sees" (592b). In the myth of Er which follows, and again in *Timaeus,* Plato's planetary model actually uses the patterns of figure 11. Throughout the *Republic,* whether jesting or serious, Socrates remains faithful to the "habitual irony" which he attributes to himself at the very beginning (337a). The sublime and the ridiculous go hand in hand as Plato proves himself skilled simultaneously in both "tragedy and comedy."

4
The Myth of Er
(REPUBLIC)

The *Republic* concludes with the tale of a “strong man, Er, son of Armenius,” full of allusions to numerical, temporal, tonal, and spatial relations. I submit that the “plain of truth” from which we start our journey to heaven is the musical proportion 6:8::9:12, present in every Platonic construction; that the static model for the planetary system (which Plato will “set in motion” arithmetically in the *Timaeus* allegory that follows) is the Dorian tetrachord and its reciprocal, constructed to heal the tyrant's suffering (fig. 11); and that Plato's “tempering” of the solar system correlates with the fundamental issue of musical temperament, $3^p5^q \neq 2^r$, dramatized in the marriage and tyrant allegories. The plausibility of this assumption is supported by the preceding mathematical material in the *Republic* and the subsequent material in *Timaeus.*

Two main sections of Er's tale contain the kind of explicit mathematical detail required for a tonal analysis and interpretation: 1) his journey to heaven, and 2) his description of the planetary system, “the spindle of Necessity.” We shall deal with these two sections separately.

ER'S JOURNEY TO HEAVEN

> *Once upon a time he died in war; and on the tenth day, when the corpses, already decayed, were picked up, he was picked up in a good state of preservation. Having been brought home, he was about to be buried on the twelfth day; as he was lying on the pyre, he came back to life, and, come back to life, he told what he saw in the other world. He said that when his soul departed, it made a journey in the company of many, and they came to a certain demonic place, where there were two openings in the earth next to one another, and, again, two in the heaven, above and opposite the others. Between them sat judges who, when they had passed judgment, told the just to continue their journey to the right and upward, through the heaven; and they attached signs*

of the judgments in front of them. The unjust they told to continue their journey to the left and down, and they had behind them signs of everything they had done. And when he himself came forward, they aid that he had to become a messenger to human beings of the things here, and they told him to listen and look at everything in the place. He saw there, at one of the openings of both heaven and earth, the souls going away when judgment had been passed on them. As to the two openings, souls out of the earth, full of dirt and dust, came up from one of them; and down from the other came other souls, pure from heaven. . . . And they told their stories to one another, the ones lamenting and crying, remembering how much and what sort of things they had suffered and seen in the journey under the earth the journey lasts a thousand years and those from heaven, in their turn, told of the inconceivable beauty of the experiences and the sights there. Now to go through the many things would take a long time, ...

When each group had spent seven days in the plain, on the eighth they were made to depart from there and continue their journey. In four days they arrived at a place from which they could see a straight light, like a column, stretched from above through all of heaven and earth, most of all resembling the rainbow but brighter and purer. They came to it after having moved forward a day's journey. And there, at the middle of the light, they saw the extremities of its bonds stretched from the heaven; for this light is that which binds heaven, like the undergirders of triremes, thus holding the entire revolution together. From the extremities stretched the spindle of Necessity, by which all he revolutions are turned.

(614b to 616c)

The points of musical interest are: a) Er's allusion to his corpse having been picked up on the tenth day and to his rebirth on the twelfth day; b) his description of the “demonic plain” with its inverse symmetry; and c) his five-day journey to the “middle of the light” which “binds heaven like the undergirders of triremes.”

a) The allusion to 10 and 12. Ten is a number Plato treats reverently. It is the number of stones in the tetractys, the sum of the three dimensions of experience by which “perceptibility comes to percipients,” the pattem of “three increases”:

point	o
line	o o
plane	o o o
solid	o o o o

Ten is also the limit of Plato's “form-numbers,” and he specifically uses it as a “time-factor.”[1] Ten is the age of the oldest child a philosophical politician can use to build a utopia, 10^2 years is the normal span of a man's life, and 10^3 years is the length of time between man's reincarnations.

Twelve for Plato and Greek music theorists has a specific tonal meaning. The musical proportion 6:8::9:12 correlates with the fixed tones of the two tetrachords within the octave, and is the module within which arithmetic and harmonic means first display themselves together. In the *Republic,* twelve is the first number mentioned (337). In a speech referring to his own “habitual irony,” Socrates suggests that the proper explanation of 12 might not be the usual one, namely, that it is 2 × 6, or 3 × 4, or 6 × 2, or 4 × 3. Thus the question as to the true meaning of 12 frames the entire *Republic* and is answered only by the tale of Er. The prime number 11, significantly avoided here, never generates in Plato's models.

b) The demonic plain. The demonic plain is ruled by rigorous inverse symmetry. There are two openings into earth and two into heaven, “above and opposite.” From the plain, the just men travel to the right and upward with signs of their judgment attached in front of them, and the unjust men travel to the left and downward with signs attached behind. Into the plain come souls pure from heaven along one path, and souls out of the earth, full of dirt and dust, along the other.

We may assume that Er's demonic plain is defined by the musical proportion 6:8::9:12, which functions as the fixed “plain or truth” throughout Plato's mathematical allegories.[2] Er's seven days in the plain correspond to the seven intervals in the octave scale formed by the variable tones together with the fixed tones. His company is required to “depart from there and continue the journey” on the eighth day when the octave limit is reached. The two paths between heaven and earth, one leading up and the other down, refer to the inverse meanings of every tone-number. They “wax” as integers and “wane” as inverse fractions. Upward means toward the generator One (Fig. 12). Figure 13 is a representation of the tonal meanings of these “paths.” Socrates views his numbers in a circle, that is, in one model octave within which inverses (Aristotle's “great and small”) are “all mixed up together” (Fig. 14).

The interpretation of the demonic plain as the muscial proportion 6:8::9:12 shows why the journey from this place to the middle of the heavens required exactly five days and why the two openings in the heaven are “above and opposite” those in earth (Fig. 15).

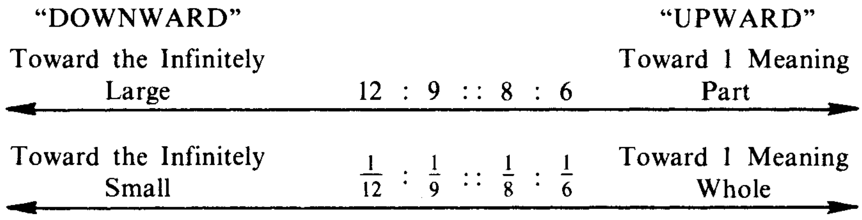


FIGURE 12

The Demonic Plain

The inverse meanings in Plato's numbers create two paths toward the number 1, representing god, and two paths through the larger numbers, symbolizing lower regions.

Socrates is concerned with the soul's journey "to the intelligible place," with "that ascent to what *is* which we shall truly affirm to be philosophy" (517b and 521c). In *Philebus* Plato wrote that only after we have studied "the number and the nature of the intervals" and have gained "real understanding" are we ready to turn our eyes to the One. "Reaching the One must be the last step of all" (18b). To Aristotle it seemed a contradiction for Plato to use One to mean simultaneously both whole and part, and to postulate "two infinites," one in the direction of increase and the other of decrease, particularly as he never reaches "the infinite in the direction of decrease" since the monad 1 is his limit there, while in the opposite direction—from the viewpoint of number—there is no "infinite magnitude" *(Physics* 206-207).

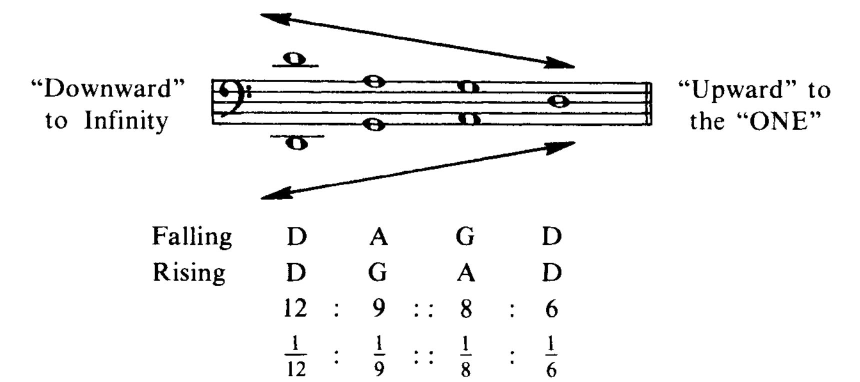


FIGURE 13

Tonal Symmetry of the Demonic Plain

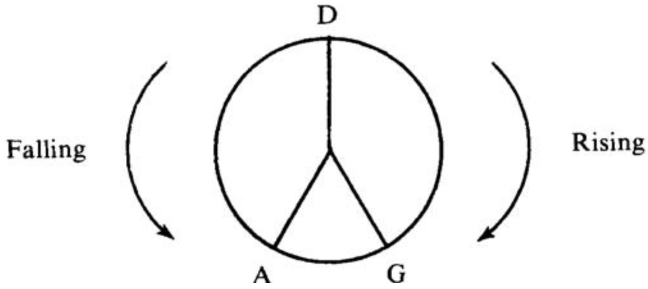


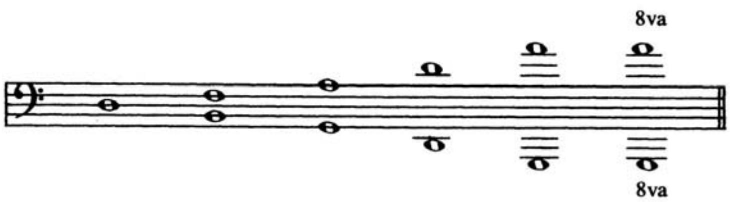
FIGURE 14

Cyclic Reduction of the Demonic Plain

ARITHMETIC INTERPRETATION

Waning →	6	:	5	:	4	:	3	:	2	:	1	= Part
Waxing →	$\frac{1}{6}$	:	$\frac{1}{5}$	:	$\frac{1}{4}$	:	$\frac{1}{3}$	:	$\frac{1}{2}$	:	$\frac{1}{1}$	= Whole

TONAL INTERPRETATION



Rising	D	f	A	D	A	A
Falling	D	b	G	D	G	G
	Minor Third	Major Third	Perfect Fourth	Perfect Fifth	Octave	
Er's "Days"	8th	9th	10th	11th	12th	

FIGURE 15

Er's Five-Day Journey to Heaven

Since up and down mean "toward the center" and "away from the center," there are two tonal paths to heaven if we "ascend" through the smaller numbers from the "plain of truth" 6:8::9:12, and they lead to A and to G when D is used as the reference tone. Er reports one-way traffic on these routes.

The language in the myth of Er, however, indicates that Plato fully understood and accepted the consequences of his position, even to postulating two different “openings” in heaven (A and G in figures 13, 14, and 15 above). Plato here chose to dramatize the inherent contradiction in the meaning of One which arises in ratio theory, and the related paradox in music theory, when the proportion 6:8::9:12 is taken as the “starting point” and numbers are interpreted simultaneously as multiples and submultipies. Aristotle actually sounds grateful for his own experiences along Plato's two roads: “Plato . . . was right in . . . asking, as he used to do, ‘are we on the way from or to the first principles?’ “ *(Nicomachean Ethics* 1095a).

c) The fiveday journey. Er's party was made to depart on the eighth day, at the octave D = 6, so that after a journey of four days through the minor third 6:5, the major third 5:4, the perfect fourth 4:3, and the perfect fifth 3:2, they arrived on the eleventh day at a place where nothing lay ahead of them but the octave 2:1. Er describes the octave double as “a straight light, like a column, stretched above through all of heaven and earth, most of all resembling the rainbow but brighter and purer.” One day later, however, on the twelfth day, having arrived at the One itself, at the transformation point between multiplication and division, between “increase and decrease,” Er sees that light now as circular. It thus relates to the cyclic octave implications of the numbers 2^n.[3] The octave—the ratio that binds the tones of the scale—appears in Plato's words as a light which binds heaven, like the undergirders of triremes, thus holding the entire revolution together.”

“Binding the city together,” Socrates says, is the true concern of law (520a). The Athenian who speaks for Plato in *Laws* declares that “the state is just like a ship at sea” (758). Later he expands the image:

> *When a shipwright is starting to build a boat, the first thing he does is to lay down the keel as a foundation and as a general indication of the shape. I have a feeling my own procedure now is exactly analogous... I really am trying to 'lay down the… character keel' we need to lay if we are going to sail through this voyage of life successfully (803).*

It is the octave that, functioning like the undergirders of triremes, binds together the entire revolution of Plato's Pythagorean heaven. His pun on *tropideia* (keel) and *tropoi* (habits or character) is also a musical pun on the Greek root of *trope* (a “turning”), a technical term in mediaeval poetry and music.

THE PLANETARY SYSTEM

From the extremities [of the light] stretched the spindle of Necessity, by which all the revolutions are turned. Its stem and hook are of adamant, and its whorl is a mixture of this and other kinds. The nature of the whorl is like this: its shape is like those we have here; but, from what he said, it must be conceived as if in one great hollow whorl, completely scooped out, lay another like it, but smaller, fitting into each other as bowls fit into each other; and there is a third one like these and a fourth, and four others. For there are eight whorls in all, lying in one another with their rims showing as circles from above, while from the hack they form one continuous whorl around the stem, which is driven right through the middle of the eighth. Now the circle formed by the lip of the first and outermost whorl is the broadest; that of the sixth, second; that of the fourth, third; that of the eighth, fourth; that of the seventh, fifth; that of the fifth, sixth; that of the third, seventh; and that of the second, eighth. And the lip of the largest whorl is multicolored; that of the seventh, brightest; that of the eighth gets its color from the seventh's shining on it; that of the second and fifth are like each other, yellower than these others; the third has the whitest color; the fourth is reddish; and the sixth is second in whiteness. The whole spindle is turned in a circle with the same motion, but within the revolving whole the seven inner circles revolve gently in the opposite direction from the whole; of them the eighth goes most quickly, second and together with one another are the seventh, sixth and fifth. Third in swiftness, as it looked to them, the fourth circled about; fourth, the third, and fifth, the second. And the spindle turned in the lap of Necessity. Above, on each of its circles, is perched a Siren, accompanying its revolution, uttering a single sound, one note; from all eight is produced the accord of a single harmony. Three others are seated round about at equal distances, each on a throne. Daughters of Necessity, Fates Lachesis, Clotho, and Atropos clad in white with wreaths on their heads, they sing to the Sirens' harmony, Lachesis of what has been, Clotho of what is, and Atropos of what is going to be. And Clotho puts her right hand to the outer revolution of the spindle and joins in turning it, ceasing from time to time; and Atropos with her left hand does the same to the inner ones; but Lachesis puts one hand to one and the other hand to the other, each in turn (616c to 617c).

There are correlations between the musical scale and the nested whorls in the spindle of Necessity with respect to d) the shape of the whorls, e) the width of the rims, 0 the pattern of colors, g) the speeds, h) the song of the Sirens, and i) the function of the Fates.

(d) The shape of the whorls. Plato now fuses the imagery of the scale, the ship, and the rainbow with a metaphor borrowed from spinning—a spindle of which the nested whorls hold the various threads with which the Fates spin our destiny. The whorls are "scooped out . . . fitting into each other as bowls fit into each other . . . lying in one another with their rims showing as circles from above." [4] A musical parallel to this imagery is our view of the tones as they lie along the string of the monochord. In figure 16 below, the falling Dorian tetrachord is paired with its opposite. This is the pattern "laid up in heaven," I believe, which Socrates points out to Glaucon and Adeimantus, Plato's older brothers, at the conclusion of the tyrant's allegory. The shaft of Necessity's spindle at the center corresponds to the point of origin of the string, our zero which the Greeks regarded as nonbeing. The radii of the various circles correspond to the string lengths of the various tones, all string lengths for the higher tones being encompassed by the fundamental length which sounds the lowest tone. This largest circle symbolizes the position of the fixed stars which were thought to enclose the planetary system. The center of the circles is the locus of the Earth.

(e) The width of the rims. Plato is exceedingly explicit in giving the relative measures of the rims, which refer to musical as well as planetary intervals, but we meet an interesting contradiction between the real spacing of the planets and our apparent spacing of the tones. When we number the tones in ascending order of pitch so that the fundamental string tone is "first and outermost," then spacing between each tone and its lower neighbor (on the right in figure 16) *increases* in Plato's order. Our first and outermost tone, the lower D bounding the octave, has no rim beyond it at all.

1	6	4	8	7	5	3	2
D	B♭	G	D	C	A	F♯	E

Plato, however, numbers the rims in *decreasing* order, from the broadest one down; his planetary rim widths vary in an order that is exactly *opposite* to that of the musical model. We see before us an explicit model but are asked to believe that truth contradicts appearances. The contradiction between reality and appearances is one of the great themes of the *Republic,* dramatized first in the myth of the cave (514-517) and anew here by way of a tone model. "Often contrary appearances are presented at the same time about the same things," Socrates observed in the preamble to the myth of Er. "Our soul teems with ten thousand such oppositions."

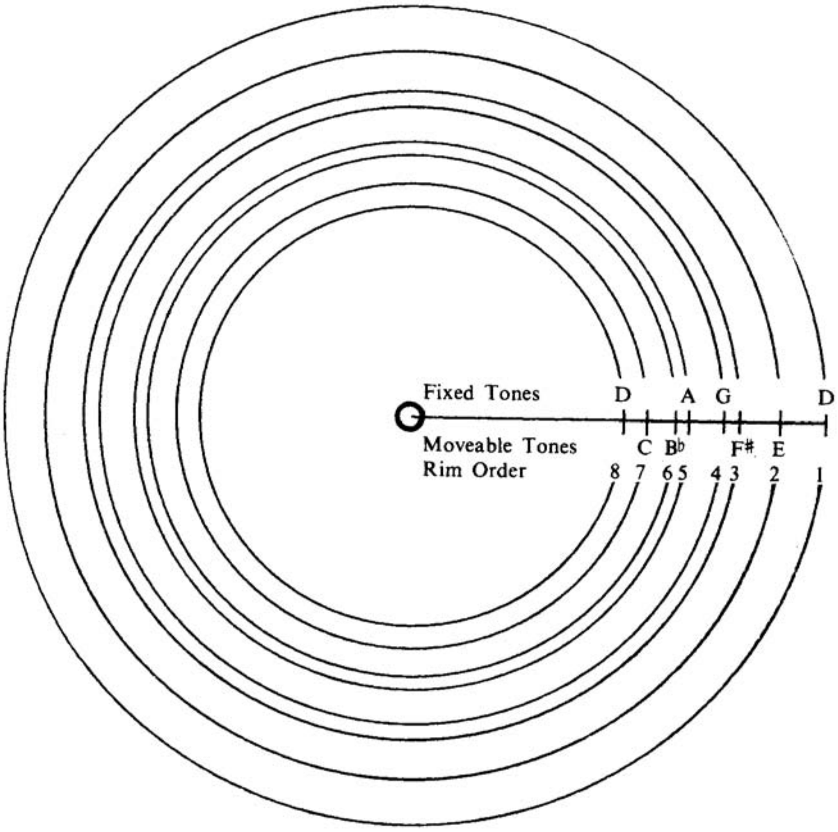


FIGURE 16

The Spindle of Necessity

The monochord string lengths associated with the tones of the Dorian tetrachord (D C B♭ A and its inverse (D E F♯ G) are treated here as the radii of Plato's nested whorls. The fundamental string length, generating the largest circle, symbolizes the circle of the fixed stars thought to enclose our planetary system. The shaft of the spindle symbolizes the point of origin of the string; it is the locus of Earth, which lies outside the planetary system. (Modeled on Brumbaugh's figure 75, page 203.)

Arithmetical Implication

5,184	5,832	6,561	6,912	7,776	8,192	9,216	10,368
D	C	B♭	A	G	F#	E	D
192	216	243	256	256	243	216	192

Dorian model (8:9) Reciprocal Dorian

But the soul rises above such convicts if it is "ruled by that which has calculated, measured, or, if you please, weighed" (602e-61 le). In *Timaeus,* which is the direct continuation of the *Republic,* Plato explains why planetary appearances in the heavens contradict the real state of affairs:

> *By reason of the movement of the Same (the great circle of fixed stars), those which revolve most quickly appeared to be overtaken by the slower, though really overtaking them. For the movement of the Same, which gives all their circles a spiral twist because they have two distinct forward motions in opposite senses, made the body which departs most slowly from itself—the swiftest of all movements—appear as keeping pace with it most closely (39).*

Here the motion of the planets is observed against the background motion of the fixed stars. The Earth, assumed to be at rest in the center of the system, provides a platform of observation from which the nearer planets seem to move faster than the distant stars; and the Moon, nearest to us, seems to move most quickly of all the bodies in the heaven. But in truth it is not so, for that distant circle of fixed stars revolves around us every day and hence must move with the greatest speed. The visible facts are contradicted by a higher truth; our monochord fits planetary *appearances,* not *reality.*

Plato's planetary theories hold little interest for us today. My point here is to demonstrate that the two planetary systems he describes in the *Republic* and *Timaeus* are self-consistent; that both are constructed from musical models and from numbers used according to the ratio theory of his day; and that he remains consistent in his postulate of a contradiction between reality and appearances in a way that forces us to abandon reliance upon models and to think in more abstract terms. In contrast to Cornford's theory that Plato was actually describing an unworkable armillary sphere, I suggest that Plato was studying the scale, quite possibly marking it off on a monochord before him, and that his imagination was rooted securely in his musical model and in his awareness that planetary orbits were more subtle than the eye knew.

f) The pattern of colors. Platonists identify the planets by the colors ascribed to them. If the heavens are represented by the monochord scale, the outermost, all-encompassing, multicolored whorl is the circle of the fixed stars and associated with the lower of the two octave tones. The seventh and brightest whorl then symbolizes the Sun and is associated with C, the next to highest tone. The eighth whorl, the Moon at the upper limit of the octave D', gets its light by reflection, an allusion to Anaxagoras' then recent *discovery* that the moon reflects the light of the sun. Plato describes the second and fifth

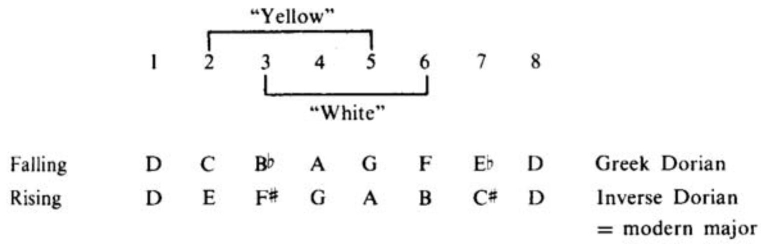


FIGURE 17

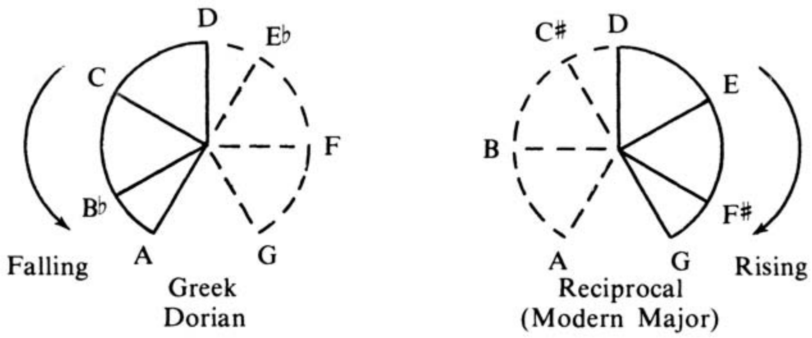
Coordination of Colors and Tones

whorls as yellow, and the third and sixth as white, so that in a general way similar colors are arranged in perfect fourths like the tones of the scale (Fig. 17).

(g) The speeds. According to Plato, the eighth whorl, here associated with the highest tone of the octave, "goes most quickly." The order in which he mentions the others—7, 6, 5, 4, 3, 2—correlates with the vibration rates of the tones. We remember that Plato considers these speeds only "apparent" and opposite to the "real" speeds. His description of the seventh, sixth, and fifth whorls as moving "together with one another" is probably not borrowed from the scale but rather refers to the periodic coordination of the relative planetary motions. The "circling about" of the fourth planet alludes to the apparent retrograde motion of Mars, which gives the system a kind of movable center of symmetry. By a loose analogy, we can identify the fourth tone of the scale as either A or G, depending on the direction taken from the starting point D; and either of these tones can function as the limit of one or the other model tetrachord.

Just when one begins to wonder whether Plato is still thinking at all of the scale, the song of the Sirens assures us that he is. We shall study astronomy, Socrates says, "by the use of problems, as in geometry, and we shall let the things in the heaven go" (530b). It seems unwise to try to interpret the myth of Er as a lecture in physics; the coping stone for Plato is dialectics, exemplified in tone.

(b)The song of the Sirens. The circles have to revolve so that the Sirens, "accompanying the revolution," will produce "the accord of a single harmony" made up of eight tones. The tone circle we have taken as a model all along, the "pattern laid up in heaven," consists of the Dorian falling tetrachord and its symmetric inverse (cf. figs. 11 and 16). If the circles revolve to the left, the falling Dorian pattern generates replicas of itself so that the Sirens sing the



Smallest Integers	384	432	486	512	576	648	729	768
Falling (Dorian)	D	C	B♭	A	G	F	E♭	D
Rising (reciprocal)	D	E	F♯	G	A	B	C♯	D

FIGURE 18

Rotation and Counter-Rotation of the Model

complete Dorian octave (D C B♭ A G F E♭ D). If the circles revolve to the right, the rising inverse Dorian pattern generates replicas of itself so that the Sirens sing the modern major scale (D E F♯ G A B C♯ D). Notice the very great reduction in numerosity associated with the tone circles of figure 18 over those of figure 16. In the *Statesman,* Plato offers the myth of a counter-rotating universe:

There is an era in which God himself assists the universe on its ways and guides it by imparting its rotation to it. There is also an era in which he releases his control. . . . Thereafter it begins to revolve in the contrary sense under its own impulse (269c).

In this dialogue, the Stranger who speaks for Plato regales young Socrates with a fantastic story concerning the reversal of the world's rotation:

All mortal beings halted on their way to bent and hoary age, and each began to grow backward, as it were, toward youth and ever greater immaturity. . . . The bodies of the young men lost the signs of manhood and, growing smaller every day and every night, they returned again to the condition of newborn children, being made like to them in mind as well as body. Next they faded into non-existence and one by one they were gone (207d, e).

In the *Statesman,* Plato elaborates on the periodic reversals between "our present life, said to be under the government of Zeus," and "the life of men under the government of Cronus" (272). These postulated reversals in the flow of time presumably affect the Siren's song so that in the age of Zeus and Plato, when music theorists took the Dorian mode as a norm, the Sirens sang the scale expected of them. In our new age of Cronus, however, when time flows in the other direction—music theory since the Renaissance is based on the opposite scale—we should probably listen for them to sing in D major. Plato's passion for abstract arithmetical and geometrical models allowed him to escape such trivial questions in the realm of appearances.[5]

Tones, rims, speeds, colors, and planets involved in the cosmology of the *Republic* are summarized in the table below; other complementary perspectives will appear in the next chapter. The tones are spaced on the monochord in the opposite order of the planets associated with them. The actual velocities (frequencies) of the tones correlate with the apparent velocities of the planets.

Rim	Tone	Width	Speed	Color	Body
8	D′	4th	1st	reflection from 7th	Moon
7	C	5th		brightest	Sun
6	Bb	2nd	2nd	2nd in whiteness	Venus
5	A	6th		yellow	Mercury
4	G	3rd	3rd	slightly ruddy	Mars
3	F♯	7th	4th	whitest	Jupiter
2	E	8th	5th	yellow	Saturn
1	D	1st		spangled	fixed stars

FIGURE 19

Coordination of Tones and Planets

i) The function of the Fates. Necessity, the goddess who rules the universe, guarding the invariant relationships between the musician's octave, the arithmetician's double, and the geometer's circle, has three daughters who help coordinate the heavenly circles. But what is wrong with the planetary system that requires Clotho to use her right hand, Atropos her left, and Lachesis each hand alternately in order to correct the velocities of the circles? The answer is supplied by musical temperament.

The three Fates must interrupt their spinning from time to time in order to temper the scale the Sirens sing. In Pythagorean tuning, and starting from D, the tones A E B F♯ C♯ are progressively too sharp, and the tones G C F B♭ E♭ are progressively too flat, to coordinate perfectly with cycles of octaves. The preceding marriage and tyrant's allegories have shown the impossibility of coordinating any cycles of intervals born of integer relations Plato cannot imagine discrepancy in heaven.

Some intimation of the necessary flaw is also projected by the loci of the thrones of Clotho, Atropos, and Lachesis. Plato describes the sisters as seated "around at equal intervals." The only equal intervals around a tone circle, that is, within an octave, are those defined by irrationals. Equal spacing of the three tones places them at $\sqrt[3]{2}$, the ratio of the equal-tempered third, somewhat larger than the Just third 4:5 but smaller than the Pythagorean third 64:81. To find an equal-tempered third on the monochord requires that we be able to construct two mean proportionals within the given limits of an octave double. This is the problem of the Delian cube, for which Plato was credited with inventing a practical solution, and for which his friend Archytas invented an elegant geometric solution.[6] The traditional mythical formulation of the problem of finding the cube root of 2 is doubling the size of the god's cubical altar" at Delphi. The loci of the musical thrones in Plato's tone-circle are suggested by figure 20: Necessity is seated at the mean on D; Lachesis—who must use both hands alternately—is shown at the equal-tempered A♭ = G♯ = $\sqrt{2}$, for which Pythagorean integer ratios show a slight excess or defect, while Clotho and Atropos, who need only one hand each, are shown at C and E, which are slightly too far to the left and to the right of D, according to Plato's ratios.

To summarize, Lachesis saves the circles from the tyrant's problem, Clotho and Atropos avoid the syntonic commas of the marriage allegory, while Necessity, with whom not even the gods contend, rules cycles as 2^p.

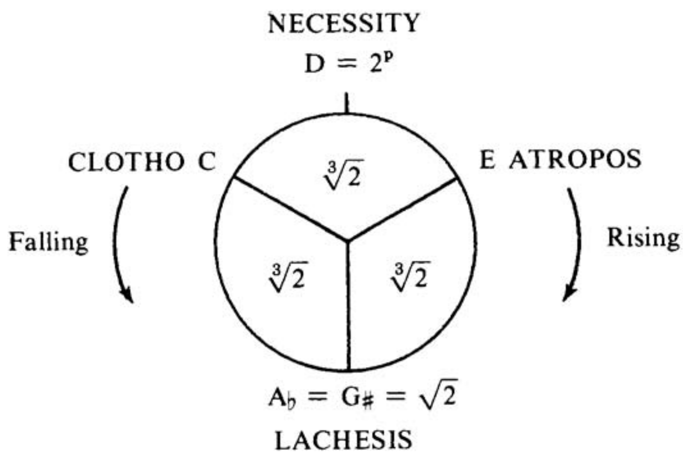


FIGURE 20

The Thrones of Necessity and the Fates

The conviction that circles are the only fit orbits for the heavenly gods, that an invariant velocity is the only kind one can decently attribute to them, and awareness that the integer relations traditionally used in Pythagorean musical arithmetic cannot possibly correlate with such heavenly cycles unless corrected—all these considerations lead with inevitable logic to the conclusion that the celestial harmony must be tuned according to equal temperament. The Just and Pythagorean tunings explored in the marriage and tyrant's allegories have proven inadequate to the task. Plato is reputedly the first philosopher to insist that number be defined broadly enough to include irrationals. His emphatic planetary theory together with his devotion to musical analogues leads directly to the acceptance of equal temperament as a celestial norm and to the evaluation of other superseded tuning systems as more or less successful approximations.

The solution offered by equal temperament meets the central concern of the *Republic,* namely, justice. At the very beginning of the dialogue, Socrates rebuts the first suggested definition, "that it is just to give to each what is owed." He rejects the subsequent definitions proposed. He finally forces his interlocuters to agree that true justice means doing what is best for the city," a task that requires men willing and able to practice moderation. This task is so difficult that the best to hope for is a pattern laid up in heaven" which will help man to found such a city "within himself." The "men" in Plato's model cities are always integers. If they do not demand that they be given exactly what is owed to them, but rather what is best for the city as a whole, then their integer values function as norms for some decent, obviously restricted, ternpered range of behavior. This is precisely the manner in which the Pythagorean musical ratios function in musical practice. We need not decide whether Plato's notion of a planetary system tempered by the Fates grew out of the musical practice of his day, or whether his grasp of the mathematical problem led to the insight that the scale has to be tempered for the coordination of divers tuning cycles. Plato saw the necessity of temperament for systems meant to function in harmony, be they musical scales, planetary orbits, or communities of just men.

5
The Creation of the or Soul
(TIMAEUS)

"One, two, three," Socrates counts, opening the *Timaeus* dialogue with the numbers which generate the World-Soul. Plato's elaborate introduction to *Timaeus* begins with a summary of the *Republic* and continues with a lengthy preview of *Critias* before Timaeus himself is allowed to speak. Socrates' tale of two cities—an ideal city limited to "essentials" (Pythagorean tuning) and enduring forever, and a luxurious city (Just tuning) "gorged with a bulky mass of things which are not in cities because of necessity" and therefore doomed—has awakened in Critias the memory of a similar tale of two cities. He tells of a war fought some 9000 years ago between Ancient Athens and the barbarian city of Atlantis. All the details have come back to him as told originally by a priest of Sais to Solon:

> *We will transfer the state you described yesterday, Socrates, and its citizens from the region of theory to concrete fact. We will take the city to be Athens and say that your imaginary citizens are those actual ancestors of ours whom the priest spoke of (26c,d).*

Before continuing with his own story, however, Critias introduces Timaeus as a man who "knows more of astronomy than the rest of us" and whose creation myth actually fills the rest of this dialogue with a monologue. "He will begin with the birth of the world and end with the nature of man," Critias announces. Plato thus tells the reader explicitly that in his own mind the ideal city of the *Republic* is exemplified by the Ancient Athens of *Critias* and that its citizens are generated in *Timaeus.* These citizens turn out to be the multiplication table for 2 × 3, hence the model for all multiplication tables.

There are two successive arithmetical sections in the *Timaeus* creation myth God first constructs an elaborate musical scale defined by numbers which function also as the "approximately 20,000" defenders of the mythical democracy of Ancient Athens, the size of the army proving to be a valuable clue to the arithmetic. God then transforms his original construction by bending it "round into a circle," and then deriving a self-perpetuating system of seven elements in which numbers are seen "in motion," as Socrates requests in a pointed allusion to his own "static" models *(Timaeus* 19b). These "moving numbers" give us still another perspective on the planetary system in the myth of Er. Plato's "World-Soul" proves to be a cosmological model in smallest integers for all possible systems definable by old-fashioned Pythagorean ratio theory (i.e., before irrationals were accepted as numbers).

Plato's Creator divides the material of primordial chaos into integer portions, reassembling it according to both the musical pattern of the Dorian tetrachord and the mathematical pattern of the three "means" (arithmetic, harmonic, and geometric) until it is "all used up (35b). He needs no fractions and does not hint at the problem of irrationals. His arithmetic belongs to the fifth century B.C., not to Plato's fourth century; it is the kind expounded again by Nicomachus in the second century A.D., meaning that it pursues those patterns of relations that are readily notated by arrays of pebbles. Before beginning the creation story, Timaeus allows himself to interpolate a review of Plato's cosmological assumptions and a disquisition on the theory of the "means" (27c-34c). Plato's creator then shows himself obsessed with circles, as befits both a musician to whom tuning systems are cyclic at the octave, and an astronomer for whom sun, moon, and planets move in perfect circles and with unvarying velocities about our stationary earth. Platonic ratio theory, music theory, political theory, and astronomy are equivalent representations of an abstract cosmological "systems" theory.

The Creator's preliminary step homogenizes his material into a mysterious blend of Sameness, Difference, and Existence before dividing "this whole into as many parts as was fitting." For musicians, tonal "existence" displays the "same" tones in "different" octaves and offers a simple example for an otherwise troublesome Platonic idea. I shall quote the two parts of the allegory separately, in Francis M. Cornford's translation, and follow each by a detailed analysis. Part I, I suggest, may have been plagiarized from Philolaus, as ancient writers claimed; it seems a tedious construction unworthy of the author of the mathematical allegories in the *Republic* and *Laws* (but reflecting Philolaus' mystical concern with Ten-ness, the "power of the Decad"). Part II, I suggest, may represent the "new math" Plato tried to introduce, a system of far greater arithmetical economy, displaying the reciprocity Aristotle alluded to in his phrase, "the Dyad of the Great and the Small" *(Physics* 203a).

PART I

THE WORLD SOUL: THE DIVISION OF SAMENESS, DIFFERENCE AND EXISTENCE

> *And he began the division in this way. First he took one portion (1) from the whole, and next a portion (2) double of this; the third (3) half as much again as the second, and three times the first; the fourth (4) double of the second; the fifth (9) three times the third; the sixth (8) eight times the first; and the seventh (27) twenty-seven times the first. Next, he went on to fill up both the double and the triple intervals, cutting off yet more parts from the original mixture and placing them between the terms, so that within each interval there were two means, the one (harmonic) exceeding the one extreme and being exceeded by the other by the same fraction of the extremes, the other (arithmetic) exceeding the one extreme by the same number whereby it was exceeded by the other.*
>
> *These links gave rise to intervals of 3/2 and 4/3 and 9/8 within the original intervals.*
>
> *And he went on to fill up all the intervals of 4/3 (i.e. fourths) with the interval 9/8 (the wholetone), leaving over in each a fraction. This remaining interval of the fraction had its terms in the numerical proportion of 256 to 243 (semitone).*
>
> *By this time the mixture from which he was cutting off these portions was all used up (35b-36b).*

The initial construction moves through three stages: a) taking the portions of 1, 2, 3, 4, 9, 8, and 27 units; b) inserting the arithmetic and harmonic means; and c) filling up the perfect fourths 4:3 with wholetones 9:8.

(a) Taking the portions of 1, 2, 3, 4, 9, 8, and 27 units. Plato's portions belong to two point-line-plane-solid progressions developed from the prime numbers 2 and 3. Powers of 2 1:2:4:8) define three consecutive musical octaves. Powers of 3 (= 1:3:9:27) define three consecutive musical twelfths, which extend through the range of four octaves plus a major sixth.[1] We shall identify 1 with D, the center of symmetry in our modern notation, and

plot Plato's numbers as ratios of both frequency and wave-length in rising-falling tonal progressions. We know from the earliest full commentary on the dialogue, written by Crantor in Plato's own century, that the Academy debated the relative merits of two different methods of "taking" Plato's proportions: a) "exponed in one row" as an arithmetic series, and b) arranged in the form of the Greek letter lambda Λ, separating the powers of 2 and 3.[2] Both arrangements, displayed in figures 21 and 22 below, lead to tonally equivalent solutions, although their arithmetic appearances are different. We shall develop the first one—"exponed in one row" in detail through all three stages before showing the simpler "lambda" arrangement of the second.

(b) Inserting the arithmetic and harmonic means.[3] We know already that the smallest integers for showing arithmetic and harmonic means within the octave 1:2 are those in the musical proportion 6:8::9:12 (chapter one). Hence Plato's "portions" must be multiplied by 6 before any of the means can be inserted. The numbers 6:8::9:12 must then be doubled and redoubled, respectively, in the second and third octaves. As a consequence, the "triple" series now begins 6:18 (i.e., = 1:3), and its means appear in the new musical proportion 6:9::12:18. These numbers are then tripled and re-tripled in successive twelfths. (Notice that 12 is the arithmetic mean between 6 and 18, the ratio 6:12 = 1:2 being an octave and the ratio 12:18 = 2:3 being a fifth; the number 9 then functions as harmonic or "sub-contrary" mean, reversing the internal order of intervals, 6:9 = 2:3, a fifth, and 9:18 = 1:2, an octave.)

(c) Filling up the perfect fourth 4:3 with wholetones 9:8. When Plato specifies that perfect fourths 4:3 are to be filled up with wholetones 9:8, leaving an undersized semitone *leimma* (literally "left-over"), he is giving us the formula for the Dorian tetrachord developed in the Tyrant's allegory (Fig. 11). The smallest integers for such a progression, as Plutarch knew, are 192 : 216 : 243 : 256; but these must be redoubled to provide integer names for *both* tetrachords of the model octave (Fig. 18). (Explanation: the second tetrachord has the ratio of the fifth 3:2 to the first, equivalent to a multiplication by 3/2, an operation which will not produce an integer if carried out on an odd number like 243, hence the need for another doubling.) Now our multipliers have increased from 6 to 192 to 384 and must be re-doubled to 768 in order to bring the very last tone-number in the Creator's original portions—namely, 27—into harmony with his tetrachord pattern. (Explanation: for the means to be inserted we had to multiply: 6 × 27 = 162. Plato's tetrachord formula requires termination by a *leimma* of 243:256; and the *least common multiple* of 162 and 256 is 20,736, exactly 768 times the original value of 27.) All of these multipliers (6, 192, 384, and 768) were known in ancient times, all are

Item	Completed progression		Falling pitch	Rising pitch	Original "portions"	"Means" inserted
1.	768		D	D	1	6
2.	864		C	E		
3.	972		B♭	F♯		
4.	1024		A	G		8
5.	1152		G	A		9
6.	1296		F	B		
7.	1458		E♭	C♯		
8.	1536		D	D	2	12
9.	1728		C	E		
10.	1944		B♭	F♯		
11.	2048		A	G		16
12.	2187		A♭	G♯		
13.	2304		G	A	3	18
14.	2592		F	B		
15.	2916		E♭	C♯		
16.	3072		D	D	4	24
17.	3456		C	E		27
18.	3888		B♭	F♯		
19.	4096		A	G		32
20.	4374		A♭	G♯		
21.	4608		G	A		36
22.	5184		F	B		
23.	5832		E♭	C♯		
24.	6144		D	D	8	48
25.	6561		D♭	D♯		
26.	6912		C	E	9	54
27.	7776		B♭	F♯		
28.	8192		A	G		
29.	8748		A♭	G♯		
30.	9216		G	A		
31.	10368		F	B		81
32.	11664	$(2^4 \cdot 3^6)$	E♭	C♯		
33.	12288	$(2^{12} \cdot 3)$	D	D		
34.	13122	$(2^1 \cdot 3^8)$	D♭	D♯		
35.	13824	$(2^9 \cdot 3^3)$	C	E		108
36.	15552	$(2^6 \cdot 3^5)$	B♭	F♯		
37.	16384	$(2^{14} \cdot 3^0)$	A	G		
38.	17496	$(2^3 \cdot 3^7)$	A♭	G♯		
39.	18432	$(2^{11} \cdot 3^2)$	G	A		
40.	19683	$(2^0 \cdot 3^9)$	G♭	A♯		
41.	20736	$(2^8 \cdot 3^4)$	F	B	27	162

FIGURE 21

Portions of the World-Soul "Exponed in One Row"

relevant to some special perspective on the scale, and my solution in figure 21 is a minor variant of that of Severus as discussed by Proclus.[4] (Severus used 768 for a multiplier, but omitted a few octave doublings and so ended up with fewer than my 41 possible tone-numbers. See Appendix I for the history of *Timaeus* solutions.) Notice that the first octave in Figure 21 shows the model diatonic scale; in later octaves, chromatic tones appear as members of overlapping tetrachords. The last octave 1:2 = 10,368:20,736 summarizes the entire tonal material, showing that the sequence consists merely of octave replications of *ten* different tone-values.

The second solution known to the Academy, the lambda arrangement, is far more elegant and instructive (see Figure 22, below). It is the outline for Nicomachus' first table of proportions, developed from the prime numbers 2 and 3 *(Introduction to Arithmetic,* Bk. II, Ch. III). Nicomachus sets the powers of 2 along one side of a triangular table, the powers of 3 along the hypotenuse, and the resultant ratios of successive perfect fifths 2:3 in the vertical columns.[5]

I have merely continued Nicomachus' table until it contains all ten of the elements Plato uses. The integer products of the prime numbers 2 and 3 are arranged in “continued geometric progression” the world's “best bonds” along three axes; octave-doubles 1:2 can be read from left to right in every row, fifths 2:3 are in every vertical column, and twelfths 1:3 lie along diagonals parallel to the hypotenuse. Furthermore, the arithmetic mean within any double can be found just *below* the smaller number, and the harmonic mean just *above* the larger number (test, for example, by looking at the loci of 6:8::9:12, the musical proportion). Similarly, the arithmetic mean within any triple (along the diagonals) lies to the *right* of the smaller number, and the harmonic mean to the *left* of the larger (test, for example, by looking at the loci of 6:9::12:18).[6] The table generates, in derivative patterns, the integers required for a model tetrachord, for a model octave, and, if continued far enough, for the entire *Timaeus* scale. Plato, with a genius for economy, failed to be very explicit about what we were to do with the creator's “portions” and thus opened a door to endless historical controversy; we have discovered that the alternatives his formula allows are all relevant in one way or another. In Nicomachus' language, the number 2^9 = 512 (in the “ninth column from unity”) “fathers” nine consecutive musical fifths which could be assigned any ten appropriate pitch names.[7] The names I have chosen show that the first three tones (A,D,G) belong to both rising and falling sequences when D is taken as the reference tone. A Greek musical theorist might have said that any tone in the table is suitable for framing tetrachords with the ratio 4:3 (i.e. resulting from fifths 2:3 being taken in each direction from the octave limits).

											frequency	wave length
1	2	4	8	16	32	64	128	256	512	=	A	G
	3	6	12	24	48	96	192	384	768	=	D	D
		9	18	36	72	144	288	576	1152	=	G	A
			27	54	108	216	432	864	1728	=	C	E
				81	162	324	648	1296	2592	=	F	B
					243	486	972	1944	3888	=	B♭	F♯
						729	1458	2916	5832	=	E♭	C♯
							2187	4374	8748	=	A♭	G♯
								6561	13122	=	D♭	D♯
									19683	=	G♭	A♯

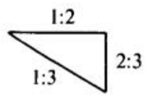


FIGURE 22

The Lambda Pattern of Nicomachus

Plato's table has been “rotated” between his own century and that of Nicomachus (c. 100 A.D.), this arrangement proving more convenient for writing. The table can be continued to infinity to contain all integer products of 2 and 3 in “continued geometric progressions” of 1:2, 2:3, and 1:3. (Compare with Crantor's lambda in figure 23.)

The functioning of every number as geometric, arithmetic, and harmonic means in variously restricted perspectives corresponds to the modern musician's statement that any tone in a sequence of fifths can function as tonic, dominant, or subdominant. Such means are unique to this table.

Notice that the counting motif, “One, two, three,” which opens the dialogue mentions the only numbers needed to generate this construction. The summary of the *Republic* which then follows emphasizes that ideal men (symbolized by odd numbers) must have wives (even numbers) and children (products of odd and even) in common. Behind Socrates' political dictum is the arithmetical fact that alternate factors of various products render “paternity” quite uncertain, while all the products together must share the same “civic” space of one “female” octave-double. When Critias breaks in with his preview of the tale of Ancient Athens and Atlantis, he encourages us to notice that the standing army of Ancient Athens—the city which exemplifies Socrates' ideal—is limited to “roughly some twenty thousand,” whereas the “World-Soul” limit is 20,736 when numbers are “exponed in one row,” or 19,683 when arranged in the triangular lambdoma. Because the smallest errors in number theory would immediately change the results by a factor of 2 or 3, the reference to “roughly some twenty thousand” seems a very strong Platonic confirmation of our construction.

Now follows a set of transformations whose musical implications have been lost over the centuries so that our interpretation breaks new ground and should therefore be considered speculative.

PART II

CONSTRUCTION OF THE CIRCLES OF THE SAME AND THE DIFFERENT

This whole fabric, then, he split lengthwise into two halves; and making the two cross one another at their centers in the form of the letter X, he bent each round into a circle and joined it up, making each meet itself and the other at a point opposite to that where they had been brought into contact.

He then comprehended them in the motion that is carried round uniformly in the same place, and made the one the outer, the other the inner circle. The outer movement he named the movement of the Same; the inner, the movement of the Different. The movement of the Same he caused to revolve to the right by way of the side; the movement of the Different to the left by way of the diagonal.

And he gave the supremacy to the revolution of the Same and uniform; for he left that single and undivided; but the inner revolution he split in six places into seven unequal circles, severally corresponding with the double and triple intervals, of each of which there were three. And he appointed that the circles should move in opposite senses to one another; while in speed three should be similar, but the other four should differ in speed from one another and from the three, though moving according to ratio (36c-d).

INTERPRETATION

Six separate actions require explanation: d) splitting the "fabric" lengthwise, e) making the two pieces cross one another in the form of the letter X, f) bending each strip "round into a circle," g) setting the two circles in motion in opposite directions, h) splitting the inner circle into seven circles, and i) establishing the relative speeds of the circles. I shall adopt the radical hypothesis that Plato is looking at patterns like those in the table of Nicomachus (Fig. 22) and asking us to pursue their implications in the wholly abstract manner he emphasized so vigorously in the *Republic.* My drawings are efforts to render visible to the reader what any acoustical theorist can imagine for himself if he merely looks at that Nicomachean multiplication table and thinks about what the numbers mean in a purely musical context.

(d) Splitting the fabric lengthwise. Since all of the numbers relevant to the Creator's scale are products of 2 and 3, I shall assume that Crantor's lambda shows us the nature of the splitting. Difference was traditionally

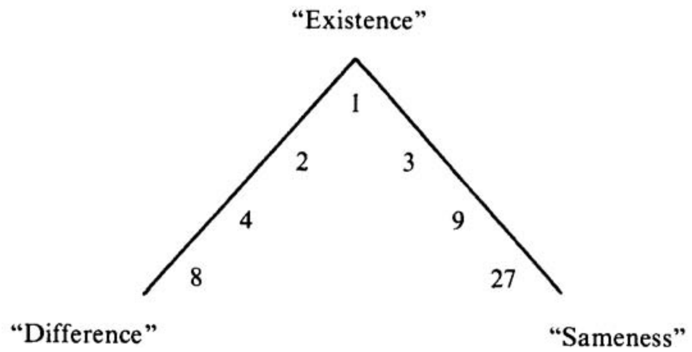


FIGURE 23

Splitting the Fabric Lengthwise

associated with numbers 2^P and Sameness with numbers 3^q. Since the material was first blended into a unity of Sameness, Difference, and Existence, one can easily sympathize with the association of numbers 2^P which differentiate octaves with "difference," and with the association of numbers 3^q which generate tone values which remain the same under octave transposition with "sameness," and to conclude that Plato is asking us to remember that every element actually participates in both series and can be looked at from either perspective.

(e) Making the two pieces cross one another in the Chi (X). Since all of Plato's numbers have reciprocal meanings—both mathematically and tonally it is easy to read Crantor's lambda as the lower half of Plato's cross, and to "reflect" its meanings in the upper half (Fig. 24).

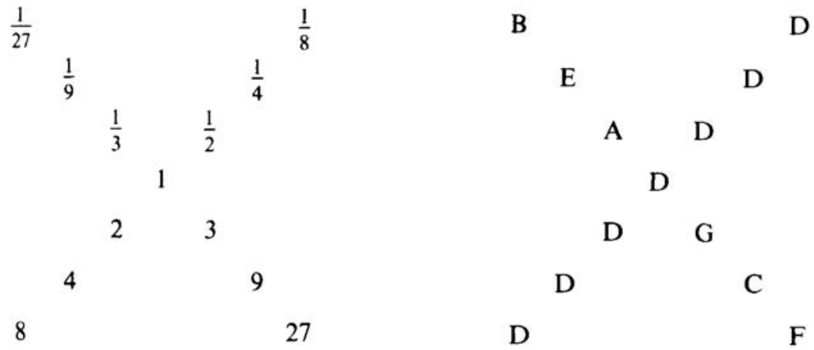


FIGURE 24

Plato's Cross (Chi = X)

(f) Bending each strip "round into a circle." In the tone circle Difference (= numbers 2^P) coincide, giving us only "cycles of barrenness," while Sameness (= numbers 3^q) remain invariant (Fig. 25).

(g) Setting the circles in motion. I assume that Plato is setting his generative numbers "in motion" to interact freely, generating "Nicomachean" tables with their reciprocal meanings, but he is also symbolizing the oblique motion of our planetary system in relation to the great circle of fixed stars, planets moving in the plane of the ecliptic and stars in the plane of the equator. In order to "see" numbers moving in this sense it helps to fill in the Crantor lambda and Plato's cross with the intervening means, and to study carefully all the directional aspects of the resulting table (Fig. 26).

Figure 26 has filled in Crantor's lambda with the means mentioned by Plato in the introduction to *Timaeus: "Solids* are always conjoined, not by one mean, but by 2" (32b). Directional aspects of figure 26 are suggested by figure 27.

Notice that one diagonal (/), the circle of the Different which differentiates octaves, includes only octave doubles. If this circle rotates to the left "by way of the diagonal," ↙ , it will include potentially the inverse meanings of 2^n to infinity, producing by itself, however, only "cycles of barrenness," to use Socrates's metaphor.

The other diagonal, however, \ , the circle of the Same, consists of the triples which define musical twelfths. If it moves to the right "by way of the side" ↳ it will move through successive intervals of the musical

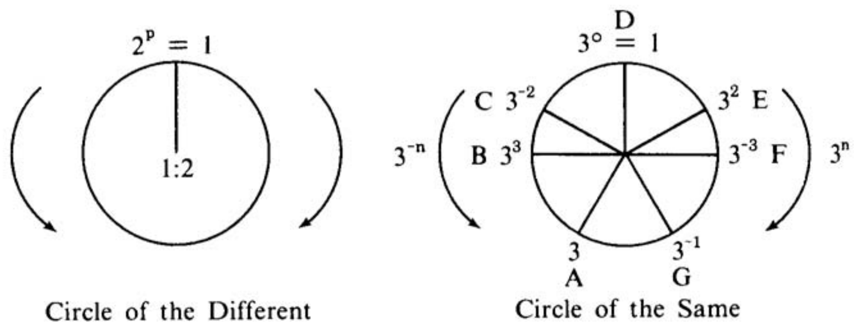


FIGURE 25

Circles of the Same and the Different

Plato's "circle of the Different"—traditionally associated with the numbers 2^p—can be thought of as the "differentiation" of octaves. His "circle of the Same"—traditionally associated with the numbers 3^q—can be thought of as showing tonal *invariance* (Sameness) within the octave treated as a cyclic module.

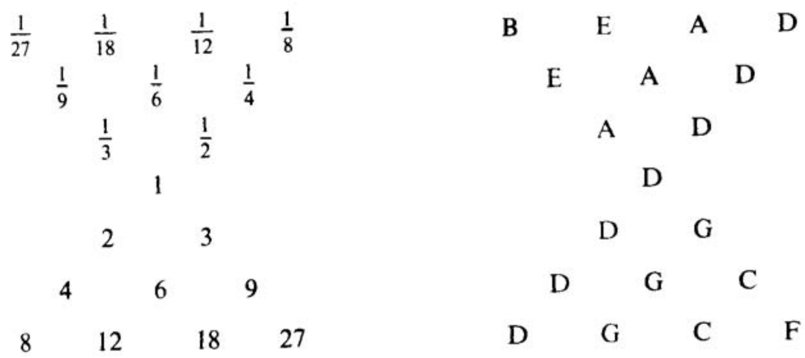


FIGURE 26

Plato's Cross With The Means Inserted

Every row of "pebbles," in any direction, is a logarithmic sequence showing the world's "best bonds.... continued geometrical proportion." (Numbers are interpreted here as multiples of string length or wave-length. If they are interpreted as multiples of frequency then upper and lower halves of the tonal interpretation are interchanged.)

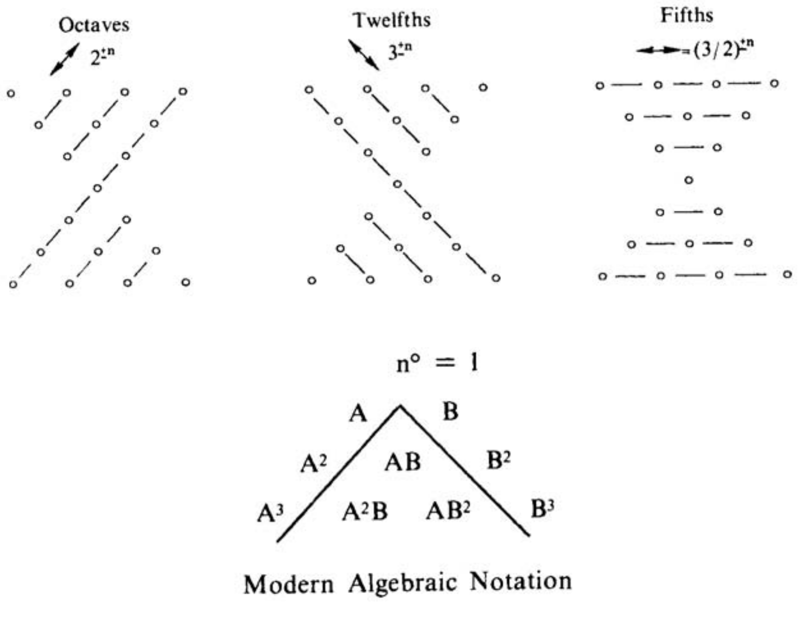


FIGURE 27

An Interpretation Of Plato's Diagonals

fifth 3:2, each displacement being an arithmetical translation (multiplication by 3/2), equivalent to a musical *transposition* by the interval of a fifth, which leaves the original pattern intact (Fig. 28).

(h) Splitting the inner circle. Plato's inner circle of the Different, /, differentiating octaves, is clearly subordinate to his circle of the Same. If he splits it "in six places into seven unequal circles," ///////, the splitting will produce, assuming it to be done "lengthwise," seven *similar* progressions of octave doubles (2^n) (See Fig. 26). That splitting can be read in any of the planimetric arrangements of the tables. The circles produced will be "unequal" in the sense that octaves of every tone are based on different, thus unequal, reference frequencies, or wavelengths, or monochord string lengths, etc., thus correlating metaphorically with the varying diameters of the planetary orbits. By arbitrarily splitting this circle into only seven parts Plato is postulating a seven-tone diatonic scale as the relevant celestial harmony. These circles correspond with the original "double and triple intervals, each of which there were three," in a double sense: each octave sequence contains the reciprocal meanings of the doubles, 1:2:4:8, and the seven different octave sequences always maintain between them the reciprocal meanings of the triple intervals, 1:3:9:27.

By splitting the inner "circle of the Different" into seven derivative circles Plato creates an invariant seven-element subset of his material. Under either rotation or counter-rotation, as in the counter-rotating universe of the *Statesman* dialogue, only seven tone-values will be present (together with their octave replications). In the scale order preferred by musicians these seven tones can be arranged in various modal configurations, the most economical arithmetically being that of

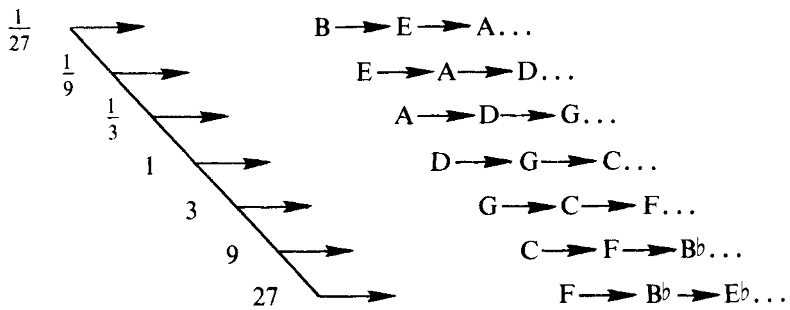


FIGURE 28

Motion of the Circle of the Same

the Dorian mode and its reciprocal in the octave 384:768.[8] It is mathematically irrelevant whether the tones be given the standard Dorian names Taylor employs, or the reciprocal (modern C major) names Cornford employs, or whether they be rearranged into the Greek Phrygian mode which maintains perfect inverse symmetry but requires slightly larger integers, or whether patterns are transposed to other keys.

Integers	384	432	486	512	576	648	729	768
falling Dorian	E	D	C	B	A	G	F	E (Taylor)
rising Reciprocal	C	D	E	F	G	A	B	C (Cornford)

Integers	432	486	512	576	648	729	768	864
falling Phrygian	D	C	B	A	G	F	E	D
rising reciprocal	D	E	F	G	A	B	C	D

Plato himself will not use such numbers; one tetrachord pattern is enough to generate both the Dorian scale and its reciprocal. The Phrygian scale is clearly derivative, requires larger integers, and is fully implied in the cube of 3 without further fuss over common denominators (1:3:9:27 means D A E B in one direction, and D G C F in the other, with octave reduction taken for granted).

(i) Relative speeds of the circles. Plato reports that in speed three of his seven circles should be "similar," and that the other four "should differ in speed from one another and from the three, though moving according to ratio." According to Greek musical theory, each octave had three fixed tones serving as tetrachord frames (notated here as D:A::G:D with the musical proportion 6:8::9:12), plus four "movable" sounds (two in each tetrachord) which had as many possible loci as Pythagorean arithmetic could provide. Plato has already shown us two different tunings for the tetrachords (framed by "the root 4:3") and will lead us to even more involved possibilities in *Laws.*

CONCLUSION

"We shall let the things in the heavens go," Socrates declared in the *Republic,* and Timaeus has followed his advice, I suggest, when giving his lecture on astronomy "by the use of problems, as in geometry," meaning the geometry of the vibrating string. In later passages Timaeus treats the creation of the world's body, and only then does he introduce the planets as instruments of time "in seven circuits seven bodies"— and associate them loosely with the seven circles of the Different moving inside the circle of the same. He mentions only Moon, Sun, Venus and Mercury by name, notes that their periods of revolution vary, and declares that the other planets have not yet been named nor their periods observed or measured “by numerical reckoning” (38-39). He breaks off this casual discussion of astronomy as being “a heavier task” than is worthwhile (38d).

It was Xenocrates, Aristotle's classmate and later head of the Academy, who insisted that Plato's World-Soul was, or was like, a “self-moving number.”[9] Aristotle ridiculed that simplistic interpretation, but musicians can be grateful for its apt descriptive power, however far it falls short of the depth of Plato's intention.[10] I have simply glossed over the additonal layers of implication Plato saw in his *Timaeus* model in order to clarify the underlying simplicity of his musical mathematics and the self-consistency of his methods.

In the myth of Ancient Athens which follows we shall discover more of Plato's own commentary on the musical World-Soul. Both the world's body and our own, he postulates in *Timaeus,* are derived from the same model as the soul and are “like it, so far as may be” (37c). There is no denigration of the body here. The physical world itself is “a perceptible god, supreme in greatness and excellence, in beauty and perfection” (92c).

What happened to that vision?

6
Ancient Athens
(CRITIAS)

In the few pages of his unfinished *Critias* dialogue Plato bequeathed us two political fairy tales and the keys to a Pythagorean poetical imagination. The cities of Athens and Atlantis are described for Socrates' entertainment; they show him, as he requested at the beginning of *Timaeus,* his ideal men and animals in motion, “actively exercising the powers promised by their form” *(Timaeus* 19c). The stories show us how two tuning systems, Pythagorean tuning and Just tuning, became the models for two cities, the first limited to “essentials” and thus destined to endure forever, and the second gorged with “luxuries” which doom it to destruction.

In the *Republic* we learn that the ideal city is a “Greek” one (470e). In *Timaeus* we learn that its name is Ancient Athens and that its citizens are generated in the way described by Timaeus, a somewhat more expert astronomer” than Socrates. Now, in *Critias,* Plato rhapsodizes mathematics as he sets the utopian city of Athens in motion. Our imagination must bridge the gap between the arithmetic of *Timaeus* and the images of *Critias* in order to understand Athens as the embodiment of the ideals of the *Republic.*

The tale of Ancient Athens is given here in A. E. Taylor's translation, severely compressed, however, to call attention to the elements capable of musical interpretation when we allow ourselves to assume that the imagery of the dialogues is actually unified in the way Plato's words suggest.

SELECTIONS FROM THE MYTH

Of old, then, the gods distributed the whole earth by regions, and that without contention....Hephaestus and Athena, as they had one nature, being brother and sister by the same father, and at one moreover, in their love of wisdom and artistry, so also obtained one lot in common, *this our land, to be a home meet for prowess and understanding. They produced from the soil a race of good men and taught them the order of their polity. . .The fighting sort had been set apart at the first by godlike men and dwelt by themselves... None of them had any private possession of his own; they looked on all things as the common store of all . . .in short, they followed all the practices we spoke of yesterday when we talked of those feigned guardians. The soil surpassed all others, which indeed, is why the district could then maintain a great army exempt from all tasks of husbandry . . . The remnant now left of it is a match for any soil in the world ... [although]what is left now is, so to say, the skeleton of a body wasted by disease . . . The soil got the benefit of the yearly `water from Zeus,' which was not lost, as it is today, by running off a barren ground to the sea . . . Thus the moisture absorbed in the higher regions percolated to the hollows and so all quarters were lavishly provided with springs and rivers.*

The Acropolis . . . was so large that it reached to the Eridanus and Illissus, enclosing the Pnyx and bounded on the side facing it by Lycabettus . . . without, directly under its slopes, were the dwellings of the craftsmen and the husbandmen who tilled the adjoining fields; higher up the fighting force had its abode by itself round the temple of Athena and Hephaestus, girdled by a single wall, like the garden of one house. On the northern side they had fashioned their common dwelling houses and winter messrooms. . . They aimed at the mean between splendor and meanness, dwelling in decent houses where they grew old, themselves and their children's children, each succeeding generation leaving them to another like itself. As for the southern side, in the summer, as was natural, they forsook their gardens, gymnasiums, and messrooms and used it for these purposes. There was only one fountain, on the site of the present Acropolis. This has been choked by the earthquake and today only shrunken rills remain in the vicinity. Then it provided all with an abundant supply of water equally wholesome in winter and summer. Such was their manner of life, then; they were at once guardians of their fellow citizens and freely followed leaders of the Hellenes at large; the number of both sexes already qualified and still qualified to bear arms they were careful to keep, as nearly as possible, always the same, roughly some twenty thousand (109b–112e).

INTERPRETATION

Eight elements in the Athens myth can be given an exact mathematical-musical interpretation: a) the divine incest of Hephaestus and Athena, by which the city is founded; b) the contiguity of the territory, “one lot in common"; c) the fighting sort "set apart"; d) we abundance of soil, lavishly provided with springs"; e) the topology of the city; f) each generation leaving "another like itself"; g) the single fountain on the Acropolis; and h) the citizens qualified to bear arms, "roughly some twenty thousand."

(a) The divine incest of Hephaestus and Athena. This city is generated by the prime numbers 2 and 3, female and male respectively, "twin children of Zeus" = 1 = geometric mean in the field of rational numbers. Such primes are "motherless" in the sense that they are divisible only by themselves or by unity, and hence are related directly to god (i.e. as a "plurality of 1's").

(b) The contiguity of the territory. The territory is one model octave 1:2 or one triangular multiplication table like those of the preceding chapter or one continuous stream of integers "exponed in one row," but Plato's notion of contiguity here must also be viewed against his perspective on the "islands of Atlantis" (cf. chapter seven) where there are gaps in the number series which define successive octave expansions.

(c) The fighting sort "set apart." Greek musical theory assumed a limitless potential subdivision of the octave, but assumed that no more than three fixed tones and four variable tones would appear together in any one *genus.* Hence the "fighting sort" I suppose to be the numbers in the musical proportions 6:8::9:12 and 6:9::12:18, the numbers in the tetrachord model which begins with 192, the numbers in the Dorian octave which begins on 384, and the numbers in the Phrygian octave which begins on 432, as shown in chapters four and five and summarized at the end of this chapter, and indeed all numbers 2^p3^q up to the limit of 10,368 required for the Spindle of figure 16, and to its double, 20,736, both World-Soul and Athenian army.

(d) The abundance of soil, "lavishly provided with springs." I visualize Plato's cross (X) as symbolizing fountains (V) fed by springs (Λ), so that Grantor's lambda (Fig. 23) is the model spring. Since the Nicomachean table in Fig. 22 is a logarithmic array, such metaphorical springs can be located anywhere within its boundaries, the metaphor being an allusion to the "translational symmetry" present in such a table.

(e) *The topology of the city.* The great size of Ancient Athens can be appreciated when we compare the large numbers employed in *Timaeus* with the *essential* numbers of the *Republic,* as I shall show in a moment. Brumbaugh interprets Plato as implying that the city is circular in shape, a notion consonant with Plato's prescriptions for other cities. The tetrachord models in figure 18 suggest why "common dwelling houses and winter messrooms" are on the "northern side" and the summer ones on the "southern side;" this inverse symmetry is a functional part of Plato's "great and small" dialectical meanings.

(f) Each generation leaving another "like itself" Multiplication by 2 and by 3 can only produce tones belonging to the *same* series of musical octaves and fifths. Hence—unlike the Atlantis model in the marriage allegory—no genetic mutation can ever occur.

(g) The single fountain on the Acropolis. In accordance with my comments under d), I suppose this to be Plato's cross in figure 24 (built from the portions of 1,2,3,4,9,8, and 27 units), together with the interpolated means of figure 26. He states the formula for this construction in *Timaeus: "*Solids are always conjoined, not by one mean, but by two" (32b). Notice that the "square" or "plane" numbers 4 and 9 are "conjoined" by the single geometric mean 6, while the "cube" or "solid" numbers 8 and 27 are conjoined by the two means 12 and 18. Such constructions, as Nicomachus notes, can be continued indefinitely. (Atlantis, with two male generators, 3 and 5, will have two such fountains on its Acropolis.)

(h) Citizens qualified to bear arms. The "roughly some twenty thousand" citizens qualified to bear arms in Ancient Athens I read as an allusion to the two alternate forms of the *Timaeus* construction. "Exponed in one row," the exact limit is 20,736; in Crantor's triangular arrays the exact limit is 3^9 = 19,683. Any error changes the result by a factor of 2 or 3, hence Plato's approximation has a far more precise meaning than appears on the surface.

CONCLUSION

It is easy to appreciate the arithmetical simplicity and tonal coherence of the *Timaeus* construction, and to see that such Athenians share wives, children and property in common, and that they live in a modest society under "self-limitations" which avoid genetic mutation and make no unreasonable demands on the number field. But Plato promised us something more. His absolutely ideal city, limited to *essentials,* would be "truly biggest" meaning most unified, and *moderately* governed—in the words of Socrates, "even if it should be made up of only one thousand defenders" *(Republic* 423). Modern Athens, he notes, has but "a small remnant" of the descendants of Ancient Athens, and possesses but a small part of the land mass the city had before the earthquakes and tidal waves which destroyed Atlantis entirely *(Timaeus* 23b—c). These comments, I suggest, point to the great economy in arithmetic achievable by adoption of Platonic double meanings in "modern fourth century Athens.

Let us review what "only one thousand defenders" of Plato's kind can do for acoustical theory in the celestial city of Callipolis (literally "beautiful city," so-named at *Republic* 527c).[1]

CALLIPOLIS

The Phrygian mode, the only one admissible besides the Dorian, needs the octave 432:864 for a pattern which remains invariant under reciprocation:

integers	432	486	512	576	648	729	768	864
falling	D	C	B	A	G	F	E	D
rising	D	E	F	G	A	B	C	D

The Dorian mode, by contrast, defines all seven of these tones plus four more with even smaller integers:

integers	384	432	486	512	576	648	729	768
falling	D	C	B♭	A	G	F	E♭	D
rising	D	E	F♯	G	A	B	C♯	D

These eleven tones are the maximum that can inhabit the chromatic octave without suffering the tyrant's problem (the Pythagorean comma which first appears at the next expansion, between A♭ and G♯). If these eleven tones are defined as powers of 3, we need only go as far as $3^5 = 243$ in each direction:

	3^0	3^1	3^2	3^3	3^4	3^5
rising	D	A	E	B	F♯	C♯
falling	D	G	C	F	B♭	E♭

If powers of 2 and 3 are developed into a Nicomachean table like that in figure 22, and reciprocal meanings are taken for granted (as both Plato and Nicomachus affirm they must be), then all eleven tones can be defined in "continued geometric progression" in the fifth column from unity, that is, between the limits of $2^5 = 32$ and $3^5 = 243$, just short of the tyrant's number.

fifths 2:3	32	48	72	108	162	243
rising	D	A	E	B	F♯	C♯
falling	D	G	C	F	B♭	E♭

The rulers, Socrates declares, are responsible for allowing his kind of city to grow "up to that point in its growth at which it's willing to be one, . . . and not beyond" *(Republic* 423b). Under Socrates' rules even the Tyrant's suffering at 729 fell within the boundary of "1000 defenders." Plato's claims are entirely appropriate when Pythagorean tuning is his model, and particularly when that model is scheduled to be tempered by the Fates. In brief, all Platonic claims about numbers are fulfilled.

Ancient Athens, despite its virtues, is a pale city compared to its rival. This myth proves to be only a musical "curtain raiser" for the tale of "barbaric" Atlantis which follows, a city even Plato found much more interesting.

7
Atlantis
(CRITIAS)

For the vivid imagery in the myth of Atlantis Plato had man possible sources. The circular cities of Ecbatana and Babylon were familiar to Herodotus (5th c. B.C.). Carthage possessed a circular naval basin, an impressive canal system, elephants, and springs, just like Atlantis. Moats, triremes, temples, huge armies and aggressive ambition were ideas familiar to Plato's audiences.[1] The interpretation being offered here, however, assumes that Atlantis is Plato's Pythagorean musical comedy, and that its serious function is to promote a deeper level of mathematical analysis of the marriage allegory of the *Republic.* I shall quote the myth in considerable detail and set its arithmetical elements alongside the facts of Plato's marriage arithmetic, letting the reader discover for himself how much can be learned by this juxtaposition. Such an adventure in musical imagination is without precedent, and can appeal to no authority but our own intuition for its verification.

Gilbert Ryle has suggested that *Critias* was written in the spring of 367 B.C. in preparation for Plato's visit with the Pythagorean community in Sicily the following year.[2] If so, the dialogue was written for the only community capable of understanding it at first hearing, and enjoyment would have been intensified by the realization that Plato was appealing to the understanding and good humor of an elite audience.

The tale of Atlantis begins in Book II of the *Republic* with Socrates' description of a second city, "luxurious ... feverish ... (and) gorged with a bulky mass of things," so that it acts as a dramatic foil for his "essential" city. In *Timaeus,* this luxurious city is named Atlantis and described as a great power which once upon a time "insolently advanced against all Europe and Asia (24e). The *Republic, Timaeus,* and *Critias* thus constitute a Platonic trilogy, unified musically, arithmetically, and politically. A. E. Taylor's translation of the myth, somewhat abbreviated in form, will be divided into four sections to facilitate analysis:

1) The genesis of Atlantis

2) The islands of Atlantis

3) The plain of Atlantis

4) The laws of Atlantis.

I shall interpret the genesis of Atlantis as beginning with the eleven tones of the Dorian scale and its reciprocal (numbers 2P3g5r ≤ 60 in figs. 5 and 6 of chapter two), reinterpreted here as “Poseidon and his five pairs of twin sons.” The islands of Atlantis will be shown as the octaves resulting from “three expansions to four limits” via 60^2, 60^3, and 60^4. The “plain of Atlantis” will be developed from the “pebble” arrays of figure 8 (products $2^p3^g5^r \leq 60^4 =$ 12,960,000, the “sovereign” number). The “laws of Atlantis” will be examined as those which establish and maintain musical order among these numbers. The metaphor of *Critias,* then, will be understood not as merely whimsical but as pedagogically motivated and, more importantly, as Plato's own mathematical prelude to his “practicable” Magnesia in *Laws.*

Part I: The Genesis of Atlantis

> *The story and a long story it is began much in this fashion. As we said before when we were speaking of the 'lots,' the gods divided the whole earth into lots, some larger, some smaller, and established their temples and sacrifices in them. Poseidon, then, thus receiving as his lot the isle of Atlantis, settled his sons by a mortal woman in a district of it which must now be described. By the sea, in the center of the island there was a plain . . . and, again, near the center of this plain a mountain. . . . In this mountain lived . . . Clito, who was just husband-high when her mother and father both died. Poseidon desired this damsel, had to do with her, and fortified the hill where she had her abode by a fence of alternate rings of sea and land, smaller and greater, one within another. He fashioned two such round wheels, as we may call them, of earth and three of sea from the very center of the island, at uniform distances, thus making the spot inaccessible to man, for there were as yet no ships and no seafaring. The island left at their center he adorned with his own hand—a light enough task for a god—causing two fountains to flow from underground springs, one warm, the other cold, and the soil to send up abundance of food plants*

of all kinds. He then begot five twin births of male offspring and divided the whole isle of Atlantis into ten parts. On the earliest-born of the first pair he bestowed their mother's dwelling place with the lot of land surrounding it, the best and largest of all, and appointed him king over his brethren. The rest he made princes, granting each of them the sovereignty over a large population and the lordship of wide lands. Further, he gave names to them all. Their king, the eldest, received a name from which the ocean, as well as the whole island, got its designation; it is called Atlantic, because the name of the first king of old times was Atlas. His younger twin brother, to whose share fell the extremity of the island off the Pillars of Heracles, fronting the region now known as Gadira, from the name of his territory, was called in Greek Eumelus, but in the language [of] his own country Gadirus, . . . All these and their descendants for many generations reigned as princes of numerous islands of the ocean besides their own, . . . (113-114)

INTERPRETATION

Six items can be interpreted musically: a) the divinity of Poseidon and the humanity of Clito; b) the rings of sea and land; c) the two springs on the central island; d) the five pairs of twin sons; e) the division of Atlantis into ten parts; and f) the identity of the first born twins, Atlas and Gadirus. (The "mountain" and "plain" will be treated later.)

a) The divinity of Poseidon and humanity of Clito. As in later Christian mythology, the divinity of a mother depends on her sons. Clito, as a mythical veil for the female prime number 2, is a "mortal woman,"—so I infer—because she produces (in the "marriage" arithmetic of chapter two) some "children of lesser birth" generated by the "human number 5," whereas Athena (cf. the myth of Ancient Athens in chapter 6) produces only sons generated by the "divine, male number 3." The "divinity" of Poseidon embraces not only his direct sons but also all of their descendants; in every row of figure 8, to be interpreted later as the "Plain of Atlantis" (cf. fig. 31), powers of the divine male number. 3 increase among successive "pebbles (counters) from 3° = 1 along the left diagonal (/), one power for each counter. The genetic defect occurs only in the vertical aspect of the table: successive sequences of fifths or fourths in the separate rows are displaced from each other by one syntonic comma 80:81 for each row in the table, a flaw which is magnified to the diesis 125:128 in four rows (cf. fig. 4c and the discussion following fig. 8). These discrepancies result from operations with the human number 5. If we regard as "best musical children" those who inhabit the central axis of figures 8 and 31 (i.e. tones in Pythagorean tuning), then we can see clearly that genetic defects can be measured by an excess or deficiency of powers of 5: in the final construction all tones in the central horizontal axis will have 5^4 as a genetic component, while the rows above will have successively more powers of 5 and those below will have successively fewer.

(b) The rings of sea and land. As Plato specifies later, there are actually three rings of sea and three of land surrounding the central island. His acoustical arithmetic in the marriage allegory displays an analogous structure: the formula "4:3 mated with 5" produces various reciprocal pentatonic and diatonic scale forms within the 30:60 octave, and three additional multiplications by 60 provide three successively larger musical octaves, i.e., "islands" in the sense that they are octave-doubles standing in bold relief against the "sea of numbers" which constitutes their background. I suggest that the numerical gaps between successively larger octave-doubles defined by successive powers of 60 (Levy's numerical "indexes") constitute Plato's intervening rings of "water." The "central island," then, presumably contains only the materials of figures 5 and 6, "divine births" generated by ratios within the perfect number 6 (i.e. , specifically excluding "unmusical grandchildren" generated by 5^2, etc.).

	(1:2)	
Poseidon and his ten sons	30:60	Central island
Children of the first generation	1,800:3,600	1st ring of land
Children of the second generation	108,000:216,000	2nd ring of land
Children of the third generation	6,480,000: 12,960,000	3rd ring of land

(c) The two springs. Athens, which had only one spring on the acropolis, had only one prime male generator, namely 3, while Atlantis, with two springs, has both 3 and 5. If figures 24 and 26 fairly represent the "spring" generated by 3, then the two Atlantean springs can be visualized as Nicomachean tables for the ratios 3:4 and 4:5. The first is relevant to the horizontal dimension of figure 8, and the second to its vertical dimension.

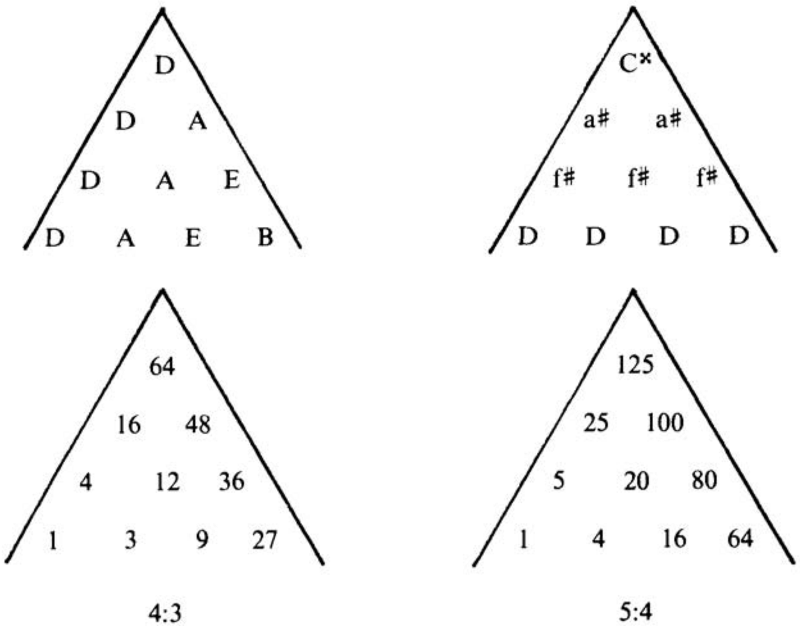


FIGURE 29

The Two Springs, 4:3 and 5:4

The ratio 3:4 generates the lateral dimension of figure 8, and 4:5 generates the vertical dimension. They are coalesced in the triangles of figure 4.

d) Poseidon and his five pairs of twin sons. The eleven tones of the Dorian scale and its reciprocal, generated by the ratio "4:3 mated with *5*" according to Plato's marriage rules (cf. fig. 5), were described as "divine" births. They can be seen as "twins" in the tone circle of figure 6: if Poseidon is the reference tone D = 1 = 3°, then his sons flank him symmetrically, c♯:e♭ c:e, b:f, b♭:f♯, and A:G. To verify arithmetically that such pairs of tones are equally spaced from the reference tone, notice that they lie within the octave 1:2 = 360:720, and that one twin must be the same interval from the lower limit that the other is from the upper limit. Hence 360:e♭::c♯:720, and all other twins similarly; since the products of means and extremes are equal in such proportions, the product of any pair of twins must be 360 × 720 = 259,200. I find it interesting that this is just ten times the number eventually taken as the "precessional" number (25,920 years) in Hindu astronomy. In *The Myth of Invariance* I suggest that this calendrical construction belonged to many ancient civilizations thousands of years older than the Greek one. Poseidon is the Greek incarnation of the

Sumerian-Babylonian Ea-Enki, and the sexagesimal arithmetic of that culture could have supplied Plato with his basic material (cf. the Historical Appendix).

Clito is invisible in our tone circle, for she coincides with Poseidon. alternatively, she represents the continuum of the octave cycle (circle) while he represents its reference limit.

(e) The division of Atlantis into ten parts. Since Poseidon and his sons now seem to divide the octave into eleven parts, we can understand Atlantis as being divided into ten parts only by assuming—along with Greek musical theorists—that the wholetone 8:9 lying in the middle of the octave (i.e., between the two tetrachords) is an "interval of disjunction" (cf. A: G in figs. 5, 6, 10, 11, 16, and 18). Plato seemingly confirms this by describing the territories assigned to Atlas and Gadirus as remote from each other, as they are when his tetrachords are made conjunct on D:

A : b♭ : b : c : c♯ : D : e♭ : e : f : f♯ : G

1 2 3 4 5 6 7 8 9 10

(f) The identity of the first born twins. It is a delightful coincidence the history of Western notation—one certainly never foreseen by Plato, who knew nothing of our present alphabetical notation—that his firstborn twins, Atlas and Gadirus, are now properly represented by A and G. (i.e. Our present center of symmetry is on D.) They are arithmetic and harmonic means within the original octave and frame the tetra-chords within which younger brothers come to birth. They appear as 8 and 9 in the musical proportion 6:8::9:12, where we seem them clearly as reciprocal meanings of Socrates' "root 4:3." The house of Atlas inherits the ruling power; Gadirus inherits "the extremity of the island," as shown above in "conjunct" arrangement. In the model octave they are multiplied by 5 (in the new musical proportion 30:40:45:60) to allow their younger brothers to have integer names:

30	32	36	40	45	48	54	60	
D	e♭	f	G	A	b♭	c	D	falling
D	c♯	b	A	G	f♯	e	D	rising

Part 11: The Islands of Atlantis

> *Now frorn Atlas sprang a prolific and illustrious house which retained the throne for many generations, . . . They possessed wealth such as had never been amassed by any royal line before them. . . . Though*

their empire brought them a great external revenue, it was the island itself which furnished the main provision for all purposes of life. . . . It also bore in its forests a generous supply of all timbers serviceable to the carpenter and builder and maintained a sufficiency of animals wild and domesticated; even elephants were plentiful. There was ample pasture for this the largest and most voracious of brutes.... So the kings employed all these gifts of the soil to construct and beautify their temples, royal residences, harbors, docks, and domain in general on the following plan.

They first bridged the rings of sea around their original home, thus making themselves a road from and to their palace . . . They began on the seaside by cutting a canal to the outermost ring, fifty stadia long, three hundred feet broad, and a hundred feet deep; the 'ring' could now be entered from the sea by this canal like a port, as the opening they had made would admit the largest of vessels. Further, at these bridges they made openings in the rings of land which separated those of water, just sufficient to admit the passage of a single trireme, and covered the openings in so that the voyage through them became subterranean, for the banks of the rings of earth were considerably elevated above sea level. The breadth of the largest ring of water, that to which the canal from the sea had been made, was three stadia and a half, and that of the contiguous ring of land the same. Of the second pair, the ring of water had a breadth of two stadia and that of land was once more equal in breadth to the water outside it; the land which immediately surrounded the central islet was in breadth one stadium; the islet on which the palace stood had a diameter of five stadia. So they enclosed this islet with the rings and bridge, which had a breadth of a hundred feet, completely by a stone wall, building towers and gates on the bridges at either end of each passage for the seawater....

Within the acropolis was the palace with the following design. In the very center, surrounded by a golden railing, which it was forbidden to enter, was an untrodden sanctuary sacred to Clito and Poseidon, the very place where the race of the ten princes had been first conceived and begotten; here, too, the seasonable offerings were made yearly to each of them from all the ten lots. Poseidon himself had a temple, a stadium long and half a stadium broad, with a proportionate height, but something un-Hellenic in its aspect.... It contained golden statues of the god standing in a chariot drawn by six winged horses, and on such a scale that his head touched the roof, and of a hundred Nereids round him riding on dolphins....

Uses were found for the waters of the two springs, the cold and the warm. The supply from both was copious and the natural flavor and virtues of their waters remarkable . . . When one had passed the three outer harbors, a wall ran all round, starting at the sea, at a uniform distance of fifty stadia from the greatest ring and its harbor, returning on itself at the mouth of the canal from the sea. This wall *was completely filled by a multitude of closely set houses, and the large harbor and canal were constantly crowded by merchant vessels and their passengers arriving from all quarters, whose vast numbers occasioned incessant shouting, clamor, and general uproar, day and night (114-117).*

INTERPRETATION

Seven elements are of musical interest: g) the great wealth of “the island itself”; h) the bridges across the rings of sea; i) the canals through the rings of land; j) the dimensions of the rings of sea and land; k) the ratios of the temple, with its “six winged horses” and its “hundred Nereids riding on dolphins”; 1) the copious water supply from the two springs; and m) the sea wall “completely filled by a multitude of closely set houses.”

(g) The great wealth of “the island itself” The model octave 1:2—Plato's one “realm of perfect pleasure” (the tyrant's allegory)—is numerically capable of infinite subdivision, and therefore capable of teaching us a great wealth of information about mathematical harmonics. The final construction in the marriage allegory (fig. 8 in chapter two, correlating with figs. 30 and 31 below) involves 89 integers $2^p3^q5^r \leq 12,960,000$ correlating with 121 tones. They constitute an extensive “transposition system” for Just tuning, teaching us much about its difficulties. But the original eleven tones, with no integers larger than 60, actually provided all the material we needed for a “tonal zodiac,” enough tones, that is, for a considerable range of modal permutation.

(h) The bridges across the rings of sea. If the rings of land are the successive octave doubles generated by powers of 60 and the rings of sea are the numerical spaces between those octaves, as suggested under b) above, then multiplications by 60 constitute “bridges.”

(i) The canals through the rings of land. Since Atlantean numerical traffic is via multiplication and division by 60, bridges and canals play complementary roles to maintain direct communication between all the relevant octave-doubles. For an interesting alternative interpretation, however, look ahead to figure 30 and notice the “channel” into the

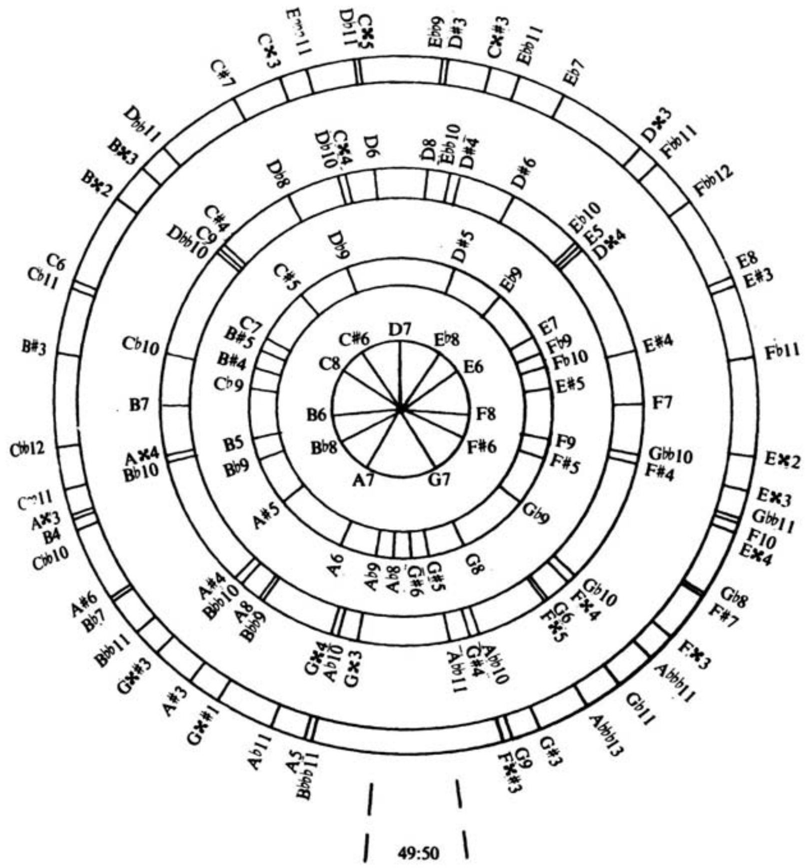


FIGURE 30

The Islands of Atlantis

Poseidon and his five pairs of twin sons are shown in the central island, and new tone-values in succeeding generations are shown in the outer rings. All radial lines should be imagined as extending into the outer ring—the fourth generation and the "sea wall" filled with confusion. Tone names are coordinated with those in the "plain" of Atlantis in figure 31, and have mainly an algebraic rather than tonal significance. This is a graph of the tonal implications of the marriage arithmetic in chapter two, successively larger octaves 1:2 being defined by successive powers of 60. Note the natural "channel" at the bottom of the circle, leading into the inner harbor through the opening formed by A♭8 and G♯6, the nearest approximations to the square root of 2.

interval of disjunction 8:9 (labelled A7:G7 in the inner circle) through the wide gap at the bottom of the outer tone circle. The narrowest gap in this channel is the *diaschisma* 2025:2048 between the tones labelled A♭8:G♯6 in the first ring surrounding the central island, which Plato may allude to when he specifies that the entrance to the inner harbor is just "sufficient to admit the passage of a single trireme." (If all of my radial "cuts" extended from the inner circles where they are "born" into the outer circle, it would be easier to see this passageway into the inner harbor, but the diagram would then be too crowded to label.) The *diaschisma* is generated by the ratio 32:45 and its reciprocal (i.e.. D:g♯::a♭:D') intrinsic to Just tuning ($45^2 = 2025$, and $2^{11} = 2048 = 2 \times 32^2$ being the nearest power of 2, the reference tone).

(j) The dimensions of the rings of sea and land. Plato's dimensions are all multiples of 10 and 60. The central island has a diameter of 5 stadia (a stadium being 600 feet), meaning 5 × 600 = 3000 feet. The inner ring of water has a breadth of one stadium (600 feet), like the ring of land around it, and Brumbaugh's translation of the numbers defining the three successive rings of water has the ratios 1:2:3, corresponding to the exponential expansion by $60^{1,2,3}$. The canal through the large outer ring is 50 stadia long (meaning 50 × 600 = 30,000 feet), 300 feet wide and 100 feet deep, truly prodigious. The bridges are all 100 feet wide (i.e., $\frac{1}{6}$th of a stadium of 600 feet) and, starting from the central island, have lengths of 600 feet, 2 × 600 = 1200 feet, and 3 × 600 = 1800 feet respectively. This spectacular engineering is presumably appropriate for a city "gorged" with luxuries.

(k) The ratios of the temple. The floor plan of Poseidon's temple is the octave 1:2 = 300:600 feet, just 100 times the essential ratio 3:6 embracing the marraige ratios 3:4:5:6. The "six winged horses" may allude to the six integers 1:2:3:4:5:6 which define everything in the allegory, and the numbers through which we ascended to heaven in the myth of Er. The "hundred Nereids riding on dolphins" I assume to be one more Platonic jest about numbers, for factors of 100 multiply everything in Atlantis to gigantic proportions. In the marriage computation, the numbers 27, 36, 48 and 60 proved to be significant numerical indices for various forms of the scale; multiplied "playfully" by 100 they became *factors* of 60^4, the "sovereign number" in Socrates' "two harmonies."

1) 12,960,000 = (36 × 100) × (36 × 100)
2) 12,960,000 = (48 × 100) × (27 × 100)

Socrates himself hinted at the roles of these hundred playful creatures:

> *Then we too must swim and try to save ourselves from the argument, hoping that some dolphin might take us on his back or for some other unusual rescue (453d).*

His own difficulty arose from his attempts to explain why women, even the old and wrinkled ones "not pleasant to the eye" should be required to exercise "naked with the men in the palaestras" (452b). Since both *even* female numbers and *odd* male numbers participate in his dialectics of opposites, Socrates was indeed in some difficulty; mathematical necessity required both sexes to have the same education in utopian politics, however strange the idea sounded to Athenians. Such sexual mixing in gymnastic would lead necessarily to "wives" being held in common. "Am I not, in your opinion, speaking of necessities?" he asks Glaucon. "Not geometrical but erotic necessities," Glaucon dryly agrees *(Republic* 458d).

(l) The copious supply of water from the two springs. Thinking of the two springs as the generative triangles of figure 29—coalesced into one in figure 4—notice in the final construction of figures 8 above and 31 below that every one of the original eleven elements in this set can act as the source for any of these constructions and their reciprocals (i.e. through three increases by the ratios 3:4 and 4:5). The result is 89 integers and 121 tone-values in just four generations, "a threat to all Europe and Asia." The whole number field is being pre-empted by this rate of geometrical expansion (i.e., × 60 for each new "generation").

(m) The sea wall. The great sea wall around the outer ring of land, "completely filled by a multitude of closely set houses," symbolizes, I suggest, the 121 tone-values in the fourth generation defined by 60^4 = 12,960,000, the integers $2^p3^q5^r$ being understood tonally and given reciprocal functions. The result is acoustical chaos: "incessant shouting, clamor, and general uproar, day and night." To help the reader *visualize* what musicians can imagine *aurally,* I have graphed in figure 30 the total of 121 tones suggested in figure 8. To simplify the graph, however, I begin with the eleven original tones (Poseidon and his five pairs of twin sons) in the central island, and then indicate only the *new* tone values in successive expansions (all of the earlier ones also recurring). The four circles can be thought of as the islands of Atlantis. (It would be too inconvenient to graph rings of sea and land with Plato's ratios, so my radial dimensions are irrelevant.)

Part III: The Atlantean Plain

I have now given you a pretty faithful report of what I once learned of the town and the old palace, and must do my best to recall the general character of the territory and its organization. To begin with, the district as a whole, so I have heard, was of great elevation and its coast precipitous, but all round the city was a plain, enclosing it and itself enclosed in turn by mountain ranges which came right down to the sea. The plain itself was smooth, level, and of a generally oblong shape; it stretched for three thousand stadia in one direction, and, at its center, for two thousand inland from the coast. . . .

Well, this plain, in consequence partly of its original structure, partly of the long-continued exertions of a succession of kings, had assumed an aspect which I shall now describe. From the first, it was naturally quadrangular, oblong, and nearly rectangular; departures from that shape had been corrected by the carrying of a fosse round it. As to the depth, breadth, and length of this fosse, it sounds incredible that any work of human hands should be so vast by comparison with other achievements of the kind, but I have to tell the tale as I heard it. It had been dug to the depth of a hundred feet, had everywhere a stadium in breadth, and, as it was carried completely round the plain, its length came to ten thousand stadia. It received the watercourses which came down from the mountains, made the tour of the plain, meeting the city in both directions, and was thence allowed to discharge into the sea. Beyond the city, straight canals of some hundred feet in width, terminated once more at the fosse on the seaside, were drawn across the plain, with a distance of a hundred stadia between every two. They were used for the floating of timber down to the town from the mountains and the conveyance by boat of natural produce generally, oblique channels of cross-communication being cut from these canals to one another and the city. There were actually two harvests in the year; in the winter the husbandman trusted to the sky for their irrigation, in the summer they looked to the earth, and released the waters of the canals. As to their numbers, each allotment of land was under an injunction to furnish one leader of a military detachment, the area of the allotment was ten stadia by ten, and the total number of these allotments mounted to sixty thousand. The number of units supplied by the mountains and the territory at large was said to be enormous, and all were regularly assigned to the different allotments and leaders according to their districts or villages. Each leader was then enjoined to furnish the army with the following contribution: one-sixth part of a war chariot, up to the full complement of ten thousand such chariots; two chargers with their riders; a pair of horses without car but supplied with a dragoon with light shield and a driver for *the pair, to stand behind the combatant; two hoplites, a pair of archers, and the same number of slingers; three light-armed throwers of stones and the same number of javelin men; four marines, up to the full complement of twelve hundred vessels. This was the war equipment of the royal city; in the other nine there were various arrangements which would take much time to describe (117-119).*

INTERPRETATION

The three elements of musical interest in this section of the myth are n) the Atlantean plain, with its network of canals, o) the two harvests, and p) the size of the armed forces.

(n) The Atlantean plain. The Atlantean plain and its network of canals, so I believe, is Plato's description of the multiplication table for numbers $3^p5^q < 60^4 = 12{,}960{,}000$ in figure 8, a table easily represented by an array of pebbles, the original Pythagorean notation, which is independent of any written language of symbols. Since the upper and lower halves of my table are arithmetically redundant, I suggest that Plato sees the lower half as his “nearly quadrangular” plain whose borders are easily straightened, visualizing the concentric circles of the capitol city as surrounding the reference tone D, set into the side of the plain, and visualizing the upper half of the table, above the central axis, as “the mountain ranges which came right down to the sea.”[3]

In figure 31 I have graphed the tone values corresponding to the integer arrays in figure 8 inside the four boundaries defined by successive powers of 60; these two figures should be studied together with figure 30. I believe Plato's radial canal system to allude to the *reflective symmetry* which every element in the table enjoys with an opposite on a line drawn directly through the reference mean on D, and that his square canal grid consists of the sequences of musical fifths 2:3 or fourths 3:4 along every horizontal line together with the chromatic semitones 24:25 which occur between successive tones in the vertical alignment. My tone names are appropriate to Plato's algebra but not to modern tuning; remember that there is a displacement by the syntonic comma 80:81 between tones sharing the same names in consecutive rows.

(o) The two harvests. Since every Platonic male odd number produces “twin sons” under his rules—equivalent to a rotation of our triangular arrays by 180 degrees—it is easy enough to think of the two harvests as being the reciprocal tones above and below the central axis of “rulers” who inhabit the only row which maintains invariance under reciprocation.

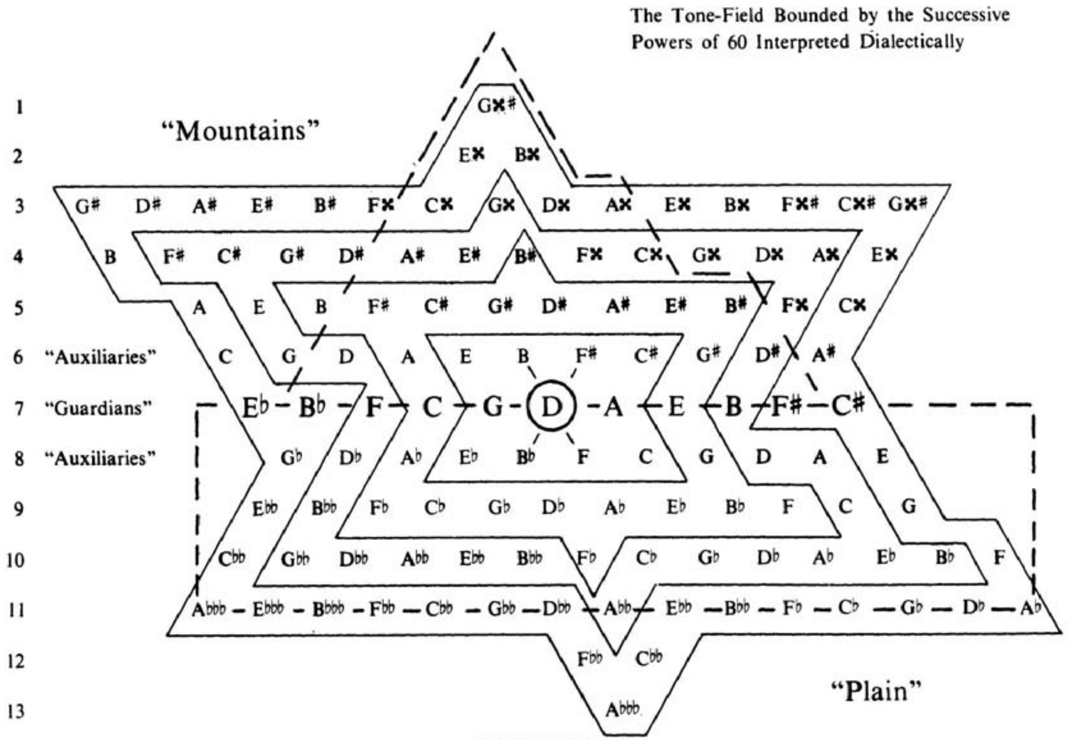


FIGURE 31

The Plain of Atlantis

The circular capitol city at D is set in the side of the plain and enclosed by the mountains. Tones are generated according to the multiplication tables of figures 4 and 8, and graphed as radial cuts in the tone-circles of figure 30.

The lower rows of the original multiplication table thus appear as a "reservoir," while the inverted table appears as a "rain cloud."

(p) The size of the armed forces. Brumbaugh analyzes the size of the armed forces as follows:

Archers		120,000
Hoplites		120,000
Slingers		120,000
Javelin-throwers		180,000
Light-armed slingers		180,000
Horsemen and charioteers		240,000
Sailors		240,000
	total	1,200,000

This is exactly 60 times the "approximately 20,000" defenders of Ancient Athens, and another hint at the Babylonian "sexagesimal arithmetic" involved here (a system on base 60 which allowed a large "1" to stand for $60^{\pm J}$, and gave every other integer the same range of meanings as multiples and submultiples of 60). By contrast to the Atlantis army, the Athenian army is limited to the central axis of figures 8 and 31.

Part IV: The Laws of Atlantis

> *The distribution of power and prerogative was, and had from the first, been this. Each of the ten kings was, in his own territory and government, supreme over persons and, for the most part, over the laws, and could chastise and put to death at his pleasure. But their authority over and intercourse with one another was regulated by the commands of Poseidon, as they were informed by the law and by an inscription left by the earliest kings on a column of orichalch preserved in the sanctuary of Poseidon in the center of the island. Here, in fact, they were accustomed to assemble at alternate intervals of four and five years, thus showing equal respect for even number and odd; in these sessions, they deliberated on their common affairs, made inquiry whether any of them were transgressing the law, and pronounced judgment . . . Judgment given, when the morning came, they wrote the judgments on a plate of gold and dedicated it and their robes for a memorial. Now there were many more special laws concerning the rights of the several kings, but the chief of these were that they should bear no arms one against another and that if any should essay to overthrow the royal house of any city, all should come to its help.... The chief command should be given to the house of* Atlas.

Also, the king should have no power over the life of any of his kinsmen, save with the approval of more than half of the ten.

Now this mighty and wondrous power, which then was in that region, the god arrayed and brought against this our own region.... For many generations, while the god's strain in them was still vigorous, they gave obedience to the laws.... They were indeed truehearted and generous.... They thought scorn of all things save virtue . . . Wealth made them not drunken with wantonness; their mastery of themselves was not lost, nor their steps made uncertain . . . Their wealth in the things of which we have told was still further increased. But when the god's part in them began to wax faint by constant crossing with much mortality, and the human temper to predominate, then they could no longer carry their fortunes, but began to behave themselves unseemly. To the seeing eye they now began to seem foul, . . . Zeus, the god of gods, who governs his kingdom by law, having the eye by which such things are seen, beheld their goodly house in its grievous plight and was minded to lay a judgment on them, that the discipline might bring them back to tune. So he gathered all the gods in his most honorable residence, even that that stands at the world's center and overlooks all that has part in becoming, and when he had gathered them there, he said ... (119-121)

INTERPRETATION

Six ideas are of musical interest: q) the authority of each king, “supreme over persons” in his own realm; r) Poseidon's requirement that the kings assemble every four and five years respectively, thus showing equal respect for even number and odd; s) the banning of warfare among the kings; t) the law giving chief command to the house of Atlas; u) the law requiring the approval of at least six kings before any subject could be given the death penalty; and v) the great virtue in the citizens at first, and their later degeneration.

(q) The authority of each king. If the ten kings are the ten tones surrounding Poseidon = 3° = 1 = reference mean, then they are musically “supreme” over any other pitches which are only a comma away, that is, too near to be recognized by the ear as different tones. Our constructions in figures 30 and 31 are full of such commas; they can be recognized visually in the tone-circles of figure 31 as radial “cuts” lying within a few degrees of Poseidon and his sons in the inner circle.

Let us agree with Aristoxenus that differences between a quarter-tone and a third-tone are aurally subliminal; that means differences of one-twelfth of a tone must be ignored. Since a wholetone is worth 200 cents in equal-temperament, and is graphed as $\frac{1}{6}$th of a circle of 360 degrees, i.e., as 60 degrees), we are agreeing that differences of 200/12 cents (≈ 17 cents) or about 5 degrees mean little or nothing to the ear in a musical context. In Platonic metaphor this means than any Atlantean king is absolute ruler over about 10 degrees of the tone circle in figure 30 i.e. about 5 degrees on each side of his own locus, or over a total range) of about 34 cents).

We do not know whether Plato actually plotted every element in figures 30 and 31. Any acoustical theorist—like Ernst Levy, for instance knows from Socrates' formula "4:3 mated with 5" that it produces all kinds of commas if continued long enough, and the tetractys patterns of paragraph c above (the two Atlantean "springs") or of figure 4 (cf. chapter two) are visibly present in a multitude of ways in figures 8 and 30, so that Plato could have pointed out his moral without resorting to the tedious construction of the detailed tables I present. But let us be precise: let us look for all numbers whose pitches lie within the subliminal areas dominated by Atlantean kings, that is, for all the kinds of *commas meaning* Diophantine approximations between numbers $2^p3^q5^r$ —generated by Socrates' formula. The *schisma, diaschisma, syntonic* comma and Pythagorean commas all lie within the narrow limits under discussion, and the *diesis* lies just beyond. Here are their sizes in ratios, cents, and degrees together with a description of their loci in figure 31, whose rows are numbered to permit easy correlation with tone names in figure 30.

The *schisma* 32768:32805, worth about 2 cents or one-half a degree (i.e., the nearest possible "conjunction") can be located, for instance, between E♭ in row 7 of figure 31 and D♯ in row 6, and between all pairs of elements with a similar planimetric spacing (i.e. between any element and one in the row above lying 8 places //////// to the right). This pair is identified as E♭7 and D♯6 in figure 30, and almost share the same radius.

The *diaschisma* 2025:2048, worth about 20 cents or 6 degrees, can be found in figure 31 between the A♭ of row 8 and the G♯ of row 6, and between all other pairs of elements with a similar planimetric spacing (i.e., between any element and one in the second row above lying 4 places //// to the right, along the diagonal).

The syntonic comma 80:81, worth about 22 cents or about 6.6 degrees, can be found in figure 31 between the reference D of row 7 and the D in row six, and between all other pairs of elements with a similar planimetric spacing (in every case they are in adjacent rows and share the same letter names, being separated also by four diagonals ////).

The Pythagorean comma 524288:531441, worth about 24 cents or 7.2

degrees, can be found between several pairs of tones *within the same row in* rows 3, 4, 10, and 11 only—when there are eleven intervening tones, as, for instance, in row 4 between B and A double sharp, and between all pairs with similar spacing.

The *diesis* 125:128, worth about 41 cents or 12 degrees, can be found in figure 31 between the reference D in row 7 and Cx in row 4 or E♭♭ in row 10, and between all pairs of elements with similar planimetric spacing (i.e. between any element and one in the third row above or below along the same diagonal (/).

These commas are of interest only to specialists in tuning theory and students of Pythagorean arithmetic. They prove the literal correctness of Socrates' claim that 60^4 = 12,960,000 is large enough to be sovereign of mathematical harmonics as it was practiced in Greece. And the "pebble" counters of figure 31, like the tone-circles of figure 30, demonstrate that the mathematical insights encoded here were available even to preliterate societies, however crude their arithmetical notation may have been.

(r) The assembly of kings. That such an assembly as we have suggested here requires "equal honor to odd and even" is obvious. But Plato's requirement that kings convene alternately in 4 and 5-year periods may have another level of meaning: Greek tuning ratios were generally limited to those known as *epimoric* ("one added"), so that 4 occurs in the ratios 4:3 and 4:5, while 5 occurs only in 4:5 and 5:6, and this mixture is required by the marriage allegory's formula, "4·3 mated with 5." It is because of this mixture that successive multiplications by 60 were required, so that the whole construction of Atlantis is at stake.

(s) The banning of warfare between kings. Our kings belong to the diatonic Dorian scale and its reciprocal; all of them together are required to establish the calendar (cf. fig. 6), and we shall meet them again *in Laws* as "boundary markers" for a radial city divided into 12 parts. They are carefully organized in symmetric tetrachords as well as in the pentatonic subsets of figure 5. Conflict between them is unnecessary, and the whole "territory" would be impoverished by the loss of any of them.

(t) Chief command to the house of Atlas. Atlas and Gadirus as arithmetic and harmonic means in the original octave are also the first incarnations of the reciprocal meanings of the divine male number 3" which generates Platonic "rulers." In Greek tuning theory the *Mese* (literally "middle") of the system is actually the *arithmetic mean* in the model octave, as well as "middle" of the

monochord string on which the system is displayed, so that it can be argued that the reference tone which I consistently label D corresponds to the *Mese* = A (Atlas?) in traditional notations of Greek tunings:

(McClain)	D		C		B♭		A		G		F		E♭		D = Dorian
		t		t		s		t		t		t		s	
(standard)	E		D		C		B		A		G		F		E
									MESE						

Enough of this, however! Let us remember that arithmetic and harmonic means play interchangeable roles in Socratic arithmetic, so that we cannot distinguish Atlas from Gadirus no matter how we approach the problem. They are identical twins. What is certain is this: *one* tone must always function as *MESE* (somewhat analogous to the requirement that "tonal" music of the 16th-19th centuries must display a definite "key" center). Plato could never have approved "atonality."

(u) The death penalty. Approval of at least six kings is required for the death penalty, not merely because six is the necessary majority among ten, but because five integers even of the best kind can define only ten tones (i.e., as, for instance, 32:36:40:45:48 in the original module 30:60). All of the pentatonic sequences of the marriage arithmetic in figure 5 also lack some of the eleven tones essential to the structure of the Atlantis model. In short, at least six integers are required to generate the essential chromatic material.

(v) The degeneration of Atlantis. In its early years Atlantis was a verritable paradise; its citizens were men of great virtue while "the god's strain in them was still vigorous," but in later generations "the human temper" (i.e. the products of 5) began to "predominate" (over the factors of 3). This was the lesson of the marriage allegory Socrates told as a "muses jest." Children would become "more unmusical," he warned. Now, in the last words of *Critias,* we hear that "Zeus, the god of gods, who governs his kingdom by law," beheld the grievous plight of the Atlanteans and was minded to lay a judgment on them, "that the discipline might *bring them back to tune."* Plato's unfinished last sentence—a sentence that could never have been finished anyway, and addressed to an audience wholly familiar with his mathematical metaphors—deserves to be studied as the best punch line a musical comedy ever had. Zeus gathered all the gods in his most honorable residence, "and when he had gathered them there, he said..."

* * *

Atlantis was a sophisticated entertainment for Pythagoreans only—if my story is “the likely one.” For the musically innocent, it is and must remain merely a Platonic fairy tale, incomprehensibly loaded with absolutely meaningless numerical detail.[4]

The last comment on this story belongs to Aristotle. According to Strabo, when asked what happened to Atlantis, Aristotle calmly replied: “Its inventor caused it to disappear.”[5]

8
Magnesia
(LAWS)

"Our songs have turned into laws!" Plato exclaims in one of his relentless puns in the dialogue *Laws,* this time on *nomoi* meaning both *laws* and *traditional melodies* for the recitation of the epics (799d). I shall take Plato seriously and interpret the hitherto baffling numerology of the dialogue—which ostensibly designs a "second best" but "practicable" city for the interior of Crete (702 and 739)—according to the same musical principles we have found at work in the *Republic, Timaeus,* and *Critias.* The absolute population limit of 5,040 "landholders" will be analyzed as the tonal "index" of a tuning system "fathered" by four primes, 2, 3, 5, and 7; the number 5040 = 2^4x 3^2 × 5 × 7 defines a tuning system like that of Plato's friend Archytas, who is the earliest theorist credited with using 7 as a tone generator. Since 5,040 is also factorial seven (7! = 1 × 2 × 3 × 4 × 5 × 6 × 7)—i.e., just 7 times larger than factorial six (6! = 720) which defines the calendar octave, or Poseidon and his ten sons (cf. fig. 6)—we have a clue as to the identity of Plato's *37* guardians, 18 from the "parent city" and 19 "new arrivals," for new arrivals among Plato's products are those generated by 7. From our arithmetical analysis of all possible products of these four prime numbers we shall discover the identities of all 37 of Plato's guardians, group them into 10-man "wrestling teams," differentiate citizens into four "property classes," draw an accurate map of his circular territory whose twelve tribes must never, under any condition, move their "boundary markers," and even learn why his supposedly practicable city of Magnesia must have so many and such strong nursemaids" for its integer children. We shall, in fact, find so much musical detail in *Laws* that we can never determine whether the slightest numerical element—and everything in Magnesia is numbered exactly—is free of musical implication.

According to Trevor J. Saunders, whose translation of *Laws* is being quoted here, the dialogue is full of "elephantine punning and other kinds of word-play, usually impossible to reproduce in English."[1] A musician cannot fail to be impressed by the rigor of Plato's legislation on music. The "Athenian Stranger" who speaks for Plato in this dialogue finds it necessary to lecture his companions on such crucial political issues as the following:

a) the three kinds of choruses proper for boys, young men, and old men (664);

b) the music for drinking parties (665-675);

c) a catalogue of musical forms: hymns, laments, paeans, dithyrambs, nomes (700);

d) musical style, and the low level of public taste (700-701);

e) the appointment of music supervisors: Supervisors of choral contests must be over 40, and supervisors of instrumental contests over 30, and choral members who fail to attend the election of a Chief Organizer are liable to fines (765);

f) the behavior of choruses at public festivals: "They swamp the holy offerings with a flood of absolute blasphemy . . . and the prize is awarded to the chorus which succeeds best in making the community burst into tears" (800);

g) standards for musical composition: "A poet should compose nothing that conflicts with society's conventional notions of justice, goodness and beauty" (801);

h) the censorship of music: official censors must be "at least fifty years of age. It would be terrible if the words failed to fit the mode, or if their metre were at odds with the beat of the music" (802);

i) the technique of lyre-playing: "Its strings must produce no notes except those of the composer of the melody. . . . The rhythms of the music of the lyre must not be tricked out with all sorts of frills and adornments" (812);

j) music education: it must be limited to three years between the ages of 13 and 16 (810);

k) composers: a composer must be "at least fifty years old, and he must not be one of those people who for all their poetical and musical competence have not a single noble or outstanding achievement to their credit" (829);

l) the funeral music suitable for a judge: fifteen girls and fifteen boys, all in white, "sing alternately a kind of hymn of praise to the dead priest" (947).

Plato's Athenian is so determined that everything in Magnesia will be standardized and regulated that “no one should dare to sing any unauthorized song, not even if it is sweeter than the hymns of Orpheus or of Thamyras” (829). It is that unflinching control over every detail in the private lives of citizens which has caused the author of *Laws* to be labelled a fascist. The possibility that he was a musical humorist has never been investigated.

About the serious intent of *Laws* there can be no doubt. At the very beginning of the dialogue Plato declares that the best legislator or judge is the one who can take a “quarreling family in hand and *reconcile* its members, without killing any of them” (627). This allusion to his unfinished story of warfare between Ancient Athens (Pythagorean tuning) and Atlantis (Just tuning) helps us to see that the even more complicated “Archytas” tuning system being developed here was, for Plato, a virtuoso example of political skill in binding radically differentiated classes into a unity.

THE DIVISIONS OF THE TERRITORY OF MAGNESIA

The legislator's first job is to locate the city as precisely as possible in the centre of the country. Next he must divide the country into twelve sections. But first he ought to reserve a sacred area for Hestia, Zeus and Athena (calling it the 'acropolis'), and enclose its boundaries; he will then divide the city itself and the whole country into twelve sections by lines radiating from this central point. The twelve sections should be made equal in the sense that a section should be smaller if the soil is good, bigger if it is poor. The legislator must then mark out five thousand and forty holdings, and further divide each into two parts; he should then make an individual holding consist of two such parts coupled so that each has a partner near the centre of the boundary of the state as the case may be . . . Finally, they must allocate the sections as twelve 'holdings' for the twelve gods . . . Again, they must divide the city into twelve sections in the same way as they divided the rest of the country; and each man should be allotted two houses, one near the centre of the state, one near the boundary. That will finish off the job of getting the state founded (745).

The best introduction to Magnesia, I believe, is Robert Brumbaugh's idealized map of the city, to which I have added tone names in figure 32. From a musical point of view, Brumbaugh's radial highways define twelve chromatic semitones in equal temperament, and his district boundaries indicate the loci of intervening quarter-tones. Regional markets and temples suggest a second,

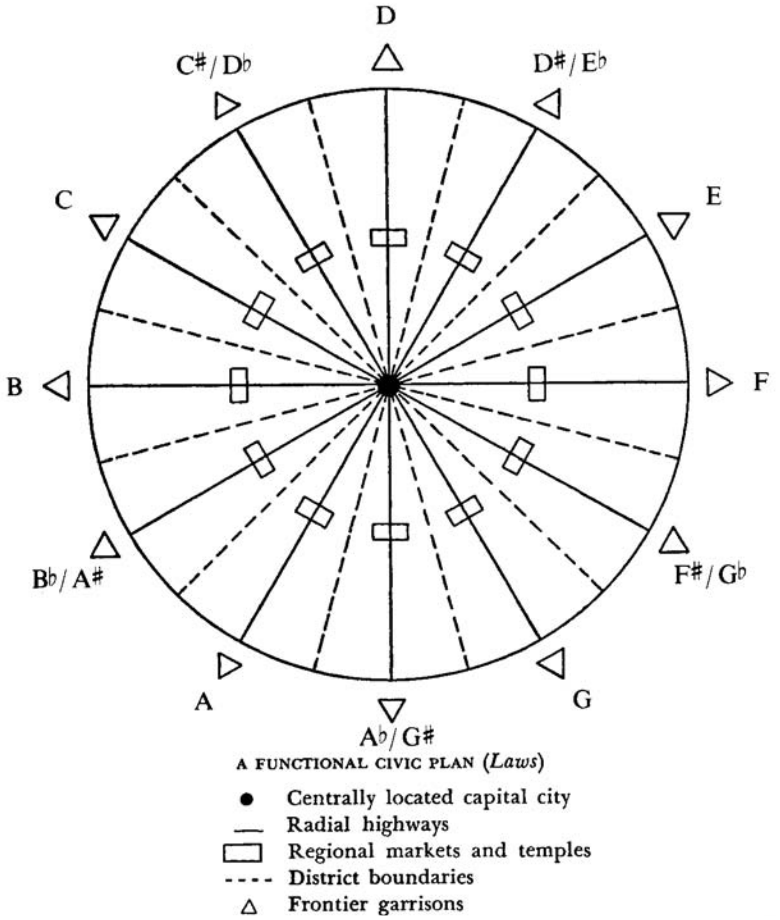


FIGURE 32

Robert Brumbaugh's Map of Magnesia

(Plato's Mathematical Imagination, p 58.)

derivative octave, needed for the Greek "Greater Perfect System." Pythagorean integer arithmetic cannot achieve such a "map." The octave 1:2 is divided into twelve equal parts by the semitone $\sqrt[12]{2}$, and into twenty four by the quartertone $\sqrt[24]{2}$. In Plato's cities, however, citizens are integers; "children . . . while they are as irrational as lines" are not allowed to rule. Since the "tribal territory" is to be divided "as exactly as possible into twelve equal sections" (760), our problem is to discover how close to Brumbaugh's idealized map—a division of the continuum of real number—we can come with rational numbers $2^p3^q5^r7^s < 5{,}040$. In short, to use the formal language of mathematics, we are faced with a problem in "Diophantine approximation."

At least eleven times in the dialogue Plato iterates that the maximum number of landholders must be rigidly maintained at 5,040, and by Glenn Morrow's count he mentions or alludes fifty or sixty times to Magnesia's thirty-seven guardians eighteen from the "parent city of Cnossos," plus nineteen more from among the "new arrivals."[2] Among integers $2^p3^q5^r7^s < 5{,}040$ there are exactly eighteen which coincide in musical meanings with citizens of Atlantis, and nineteen more "new arrivals" generated by the prime number seven which can be linked to them in the ways Plato specifies. We shall study first how eighteen "divine births" from the *Republic* and *Critias,* meaning $2^p3^q5^r$ 6! = 720, divide the circular territory into twelve approximately equal parts. Next we shall notice how nineteen "new arrivals" generated by the prime number seven make further "quarter-tone" cuts in the octave. Finally, using the Archytas tuning system as a guide, we shall test the musical utility of our Platonic guardians. And in chapter 9 we shall test their trigonometric virtuosity.

EIGHTEEN GUARDIANS FROM THE PARENT CITY

In chart 33 I have graphed the eighteen tone-values Magnesia inherits from the marriage allegory of the *Republic.* The rationale is as follows: If D at 5,040 (= $2^4 \times 3^2 \times 5 \times 7$) is the Levy "tonal index" for what I call "Archytas" tuning out of respect for Plato's friend who first integrated such numbers, then $2^4 \times 3^2 \times 5 = 720$ is the tonal index for the "Just tuning" which it includes as a subset, and $2^4 \times 3^2 = 144$ is the index for a further substratum in "Pythagorean tuning." Remembering that factors of 2 merely produce cyclic identities, we readily see that within the given index the reference mean on D can be surrounded by only two pairs of musical fourths or fifths (C G D A E in figure 33a):

	72	81	96	108	128	144
rising	D	E	G	A	C	D
falling	D	C	A	G	E	D

The pure thirds 4:5 and 5:6 which the prime number 5 "fathers" on this base can be studied from the multiplication table for integers $3^p5^q < 720$, shown arithmetically in figure 33b and tonally in 33a. The table can be

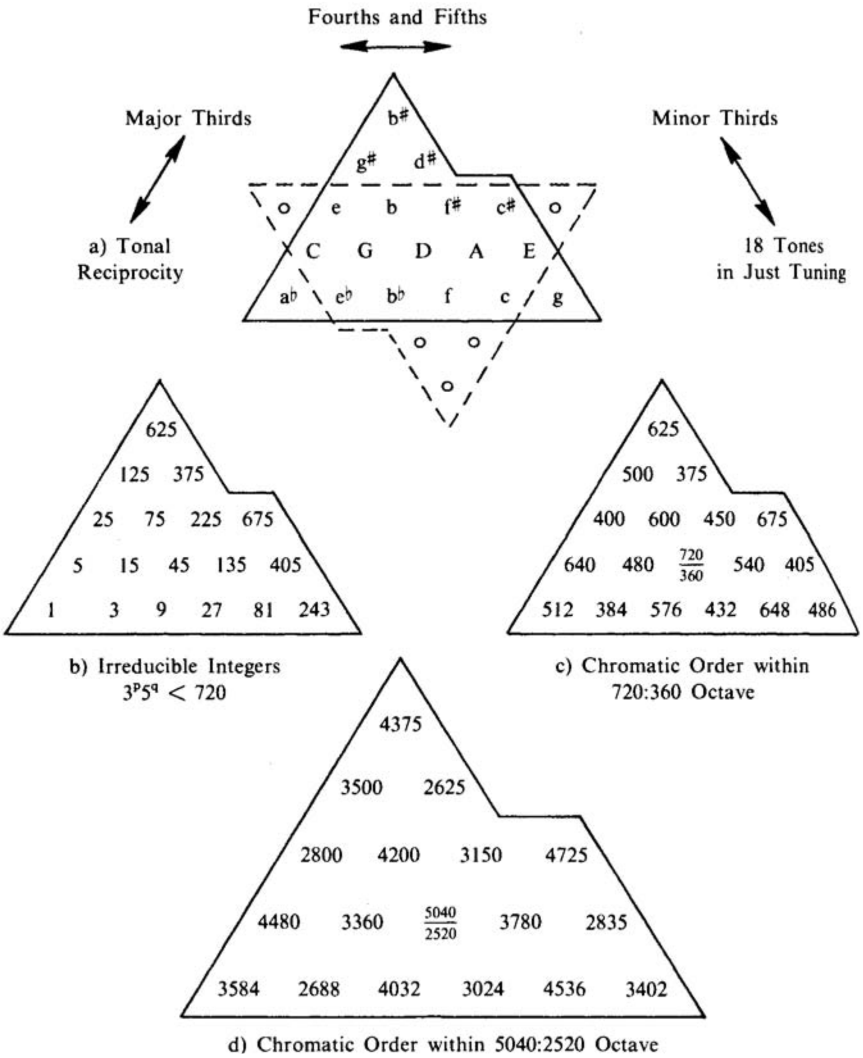


FIGURE 33

Arithmetical Computation of the Eighteen Guardians

These tables exclude octave redundancies and facilitate the study of available ratios within given limits. Within each set of tone-numbers, several pairs exhibit perfect inverse symmetry around the reference tone, D.

rotated to show which elements have "twins" within that limit, that is, reciprocals across the axes of symmetry through D, from any point within the table. Multiplication by 2^p produces the linear scale order of figure 33c within the "Just" octave 1:2 = 360:720; eleven of these elements are "Poseidon and his five pairs of twin sons" (cf. figs. 4 and 30), while seven others are relatives to whom we paid no special attention before. Further multiplication by 7 in figure *33d* shows how these eighteen elements finally look within Magnesia's own model octave 1:2 = 2,520:5,040, that is, when finally "conversable" (i.e., commensurable) with "new arrivals."

The symmetries available within this set of eighteen numbers are the best available approximations for the twelve chromatic semitones in Brumbaugh's idealized drawing (they are graphed below in figure 34). They consist of Poseidon's five pairs of twin sons plus an alternate C and E. These thirteen tones are related to five others within the index of 720. I interpret "citizens of the highest property class" to be the five tones C G D A E in Pythagorean tuning linked by *largest ratios* of the rising fifth 2:3 or falling fourth 3:4. "Citizens of the second highest property class" will then be the additional thirteen tones within the Just limits of 720, generated by the second largest ratios of the major third 4:5 and minor third 5:6.

The unit from which these numbers are generated as simple multiples is the cyclic identity, $512 = 2^9 = 2^0 3^0 5^0 = 1$, slightly askew $\sqrt{2}$ lying opposite the reference mean on D. (Remember that powers of 2 never alter the loci of tones in a tone-circle.) I have here called it *a♭* (interpreting the numbers as ratios of frequency); its reciprocal would be *g♯,* similarly displaced to the opposite side of $\sqrt{2}$ (not so far, however, as g♯ = 500, the nearest number present in this set). The 360 units within the octave 1:2 = 360:720 correspond to the 360 days in the Magnesian calendar; the gap between a♭ = 512 and its reciprocal g♯ (not shown, because no number is present in this set for it) correspond to the five extra days which are added at the end of the year for "elections," i.e., like the five extra days in the Egyptian calendar. This musical gap, called a *diaschisma,* has the ratio 2025:2048, and is worth about 20 cents, taking the octave as 1200 cents, meaning equally spaced logarithmic units. The musical shortage of $\frac{1}{60}$ of the octave of 1200 cents approximates the calendrical shortage of 5/365 = $\frac{1}{73}$. The twelve semitones correspond to the twelve months, and if the "year" is considered to start at a♭ = 512, then the error, going clockwise around the circle (cf. fig. 34), occurs at the end of the twelfth month, which Plato assigns appropriately to Pluto, god of the underworld, bidding Magnesians pay him the same reverence due to the other gods *(Laws* 828).

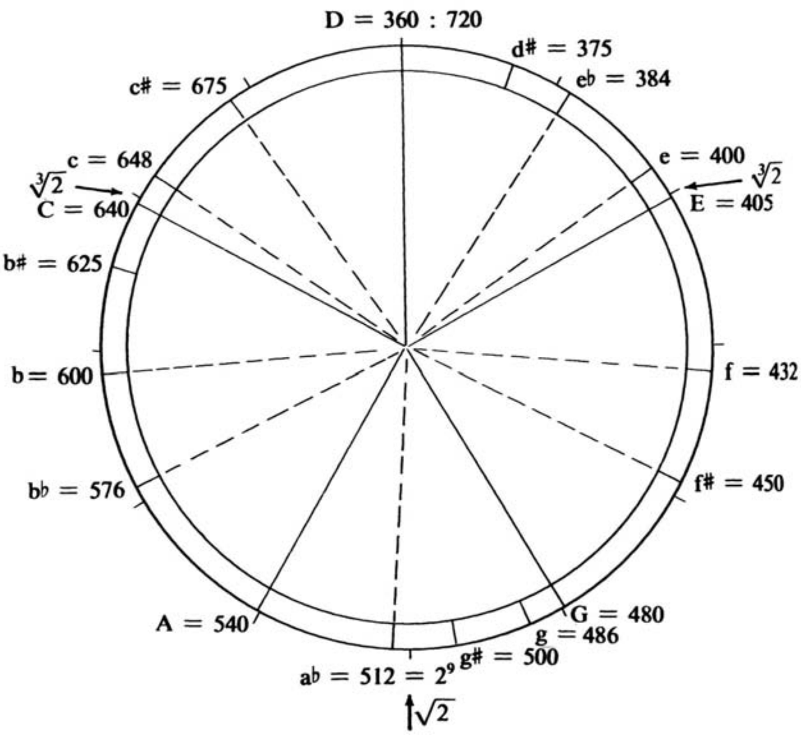


FIGURE 34

Eighteen "Parent" Guardians as a Tone-Circle

The 18 tones shown here are from the marriage allegory of the *Republic,* the arithmetical model for doomed Atlantis. They constitute the citizens of the two highest classes in Magnesia, derived from repeated operations with 3 and 5. Not all of them participate in dividing the "territory" into twelve parts, for reasons discussed in the text. The square root of 2 divides the octave into two equal parts, and the cube root of 2, taken reciprocally around the square root, correlates with the equal-tempered E and C which divide the octave into three equal parts. D^1 remains the geometric mean.

Under reciprocation, corresponding to a counterclockwise progression, the number 512 would lie about six degrees to the right of where it does now, so the error would again arise in the twelfth month, counting in the other direction.

Each of Magnesia's twelve tribes selects 60 guardsmen for a 720-man force on duty at all times (760). Her "three generals" and also three "Expounders" (of the law) correlate with the tones D, A and G in the musical proportion 6:8::9:12 which frame the tetrachords, all other tones being considered "movable sounds" in Greek musical theory. Guardsmen serve for two years, visiting Magnesia's outposts "round in a circle," being transferred each month "to the next district on the right . . . to the East," but reversing direction the second year (760). One of their functions is to ensure that "boundary markers" are never moved, and the ones shown here never are, except for the number 512, which moves back and forth a few degrees under Pluto's supervision, hemming in the square root of two (760). Notice, however, the syntonic commas 80:81 at C-c and E-e. Plato has provided for them:

> *We can see that it is not universally true that one district extends right up to the boundary of another. In some cases there is a no man's land in between, which will extend so as to touch either boundary and occupy an intermediate position between the two (878).*

These commas enclose the cube root of two when viewed from the perspective of the equal-tempered A♭ = G♯ (see the thrones of Lachesis, Clotho and Atropos in figure 20).

The 360 units in the octave and calendar correlate with the "thirty dozen" members of Magnesia's ruling council. Council members are divided into 12 groups of 30, each group presiding for a month at a time. "No less than three hundred and sixty-five" such officials, however, are chosen to perform the "daily sacrifices," thus covering the five extra days in Pluto's month (828). The 60-man squads of guardsmen and 30-man groups of council members can be read as allusions to the 30:60 octave studied in figures 4, 5, and 30, generated by the marriage formula *"4:3 mated with 5." The five* "Captains of the Guard" or "Country-Wardens" correspond to the basic pentatonic sets of figure 5 (the wedding feast," whose regulations came from *Laws* 775), or to the five "first-class" citizens C G D A E.

NINETEEN GUARDIANS FROM THE NEW ARRIVALS

The new arrivals in Magnesia's arithmetic are generated by the prime number 7 which Plato has not used before. The number 7 functions as the arithmetic mean in the musical fourth 3:4, generating the "citizens of the third highest property class" by the third largest ratios of the "septimal third" 6:7 and "septimal tone" 7:8. The "eighteen guardians from the parent city" must be multiplied by 7 (as shown in figure 33d) to make them commensurable with new arrivals. As a result, every one of the eighteen can be multiplied by $\frac{6}{7}$ and many of them can also be multiplied by $\frac{7}{6}$ so that our problem is to decide which nineteen "new arrivals" Plato means us to choose from the plethora available. Both the possible candidates and my "election of 19 are shown in detail in figure 35, but Plato's very complicated game with numbers requires careful explanation.

Plato is obviously looking for pairs of numbers which function as "twins" by making symmetric cuts on each side of his reference "mean" on D. Such cuts in the continuum of the octave 1:2 remain invariant under reciprocation. I believe the clue to his game is his requirement that during military training "we'll have our citizens fight each other *armed—man* to man, two a side, and any number per team up to ten" (833). We have already seen how Poseidon's sons constitute one such "ten-man team." Now an inspection of the numbers belonging to the "eighteen guardians, from the parent city" in figures 33 and 34 shows that exactly ten of them are divisible by 6, hence these ten can be multiplied by $\frac{7}{6}$ to produce a new ten-man team made up of "new arrivals." Furthermore, all ten possess new reciprocals around the reference mean on D, these reciprocals constituting Magnesia's third ten-man team. But there is a problem with one of these latter team members: notice that f♯ = 3200, reciprocal of b♭ = 3969, is *not* among the eighteen candidates generated by $\frac{6}{7}$ from the "parent" guardians. This citizen, though a good team member, is apparently genetically ineligible to serve among the nineteen guardians (all of whom can be generated directly from the eighteen "parent guardians" via multiplication by $\frac{6}{7}$ and $\frac{7}{6}$, or their modular equivalents $\frac{12}{7}$ and $\frac{7}{12}$).

In fairness to the reader, let me point out that we cannot determine from a simple inspection of the tables of figure 35 which new arrivals are reciprocals or twins of each other unless we can simultaneously imagine the locus of each tone in the octave and the direction of each multiplication. An arithmetic test proves easier. By way of example with the numbers 3200 and 3969, notice that 2520, the lower limit of the octave, must have the same ratio to the smaller number as the larger number has to 5040, the upper limit of the octave, for them to be Platonic twins: 2520:3200::3969:5040. Since the products of means and extremes are equal, we can multiply the octave limits—2520 × 5040 = 12,700,800—to find the product which can then be divided by *any* number within the octave to locate, as quotient, its reciprocal mean in such a proportion and its Platonic twin in the tone-circle. The products of each pair of reciprocals in the election arithmetic of figure 35 is 12,700,800, slightly less than Socrates' "sovereign" number of 60^4 = 12,960,000 in the marriage allegory of the *Republic.*

			CANDIDATES		
				TEN-MAN TEAMS	
18 "PARENTS"				THE TEN	RECIPROCALS
1.	D	5040 × 6/7	= 4320, $b^{\wedge 1}$;	× 7/12 = 2940, f^{v1};	($b^{\wedge 1}$, 4320)
2.	c♯	4725 × 6/7	= 4050, $a\sharp^{\wedge 2}$;		
3.	c	4536 × 6/7	= 3888, $a^{\wedge 2}$;	× 7/12 = 2646, $e\flat^{v2}$;	$c\sharp^{\wedge 2}$, 4800
4.	C	4480 × 6/7	= 3840, $a^{\wedge 1}$;		
5.	b♯	4375 × 6/7	= 3750 $g\times^{\wedge 2}$;		
6.	b	4200 × 6/7	= 3600, $g\sharp^{\wedge 2}$;	× 7/6 = 4900, d^{v2};	$d^{\wedge 2}$, 2592
7.	b♭	4032 × 6/7	= 3456, $g^{\wedge 2}$;	× 7/6 = 4704, $d\flat^{v2}$;	$d\sharp^{\wedge 2}$, 2700
8.	A	3780 × 6/7	= 3240, $f\sharp^{\wedge 1}$;	× 7/6 = 4410, c^{v1};	$e^{\wedge 1}$, 2880
9.	a♭	3584 × 6/7	= 3072, $f^{\wedge 2}$;		
10.	g♯	3500 × 6/7	= 3000 $e\sharp^{\wedge 2}$;		
11.	g	3402 × 6/7	= 2916, $e^{\wedge 2}$;	× 7/6 = 3969, $b\flat^{v2}$;	[$f\sharp^{\wedge 2}$, 3200]*
12.	G	3360 × 6/7	= 2880, $e^{\wedge 1}$;	× 7/6 = 3920, $b\flat^{v1}$;	$f\sharp^{\wedge 1}$, 3240
13.	f♯	3150 × 6/7	= 2700, $d\sharp^{\wedge 2}$;	× 7/6 = 3675, a^{v2};	$g^{\wedge 2}$; 3456
14.	f	3024 × 6/7	= 2592, $d^{\wedge 2}$;	× 7/6 = 3528, $a\flat^{v2}$;	$g\sharp^{\wedge 2}$; 3600
15.	E	2835 × 12/7	= 4860, $c\sharp^{\wedge 1}$;		
16.	e	2800 × 12/7	= 4800, $c\sharp^{\wedge 2}$;		
17.	e♭	2688 × 12/7	= 4608, $c^{\wedge 2}$;	× 7/6 = 3136, $g\flat^{v2}$;	$a\sharp^{\wedge 2}$, 4050
18.	d♯	2625 × 12/7	= 4500 $b\sharp^{\wedge 2}$;		

FIGURE 35

Determination of the Nineteen New Arrivals

Genetic Code: Capital letters identify "citizens of the highest property class" generated by the "divine male number 3," and small letters identify "citizens of the second highest property class" generated by the "human male number 5." "Citizens of the third highest property class" generated by 7 are marked by (v) when flatter and by (∧) when sharper than the letter notation suggests. This code facilitates the study of transposition possibilities (cf. fig. 39). An asterisk on the number 3200 indicates a "team member" ineligible to be also a "guardian." Numerical exponents indicate 1st and 2nd class parentage.

When the nineteen guardians from among the new arrivals are projected into the tone-circle of figure 36 along with the eighteen parent guardians from the Atlantis arithmetic, we discover a new link to the marriage allegory of the *Republic:* the square root of two now lies *within* the ratio 49:50 formed by two "third class" guardians, and this ratio, is flanked by another pair 48:49, three numbers Socrates emphasized in an extravagant manner. Numerical elements in the *Republic* and *Laws* thus

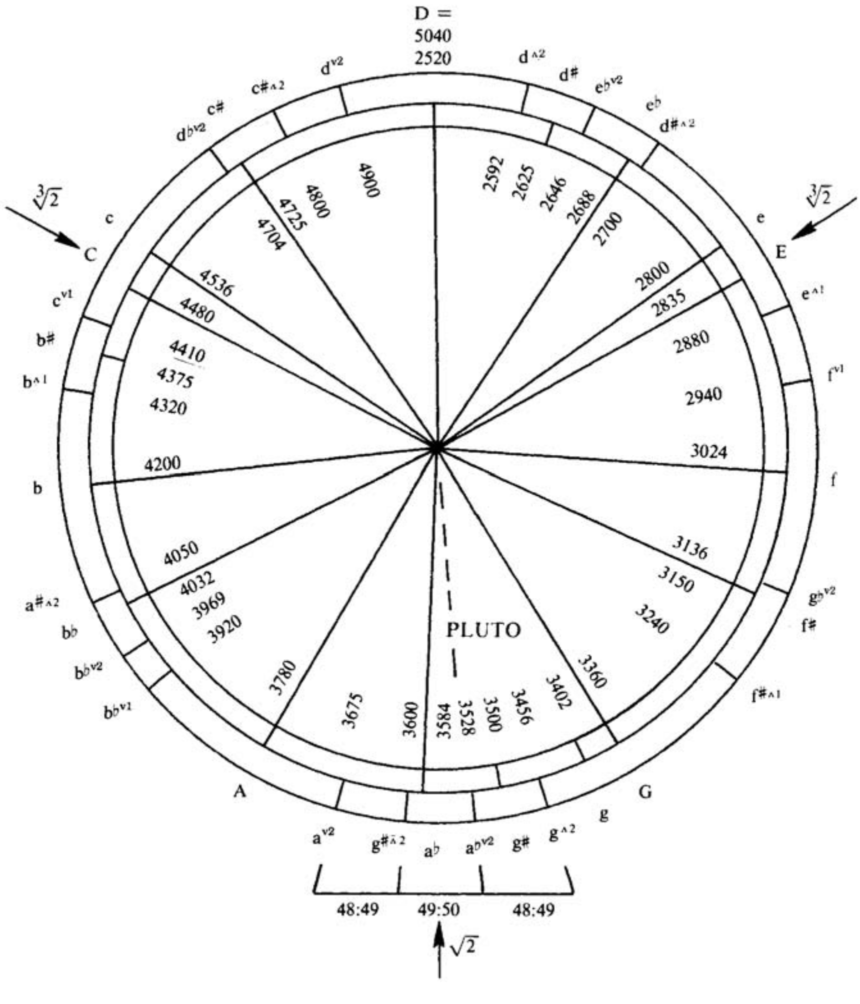


FIGURE 36

The 37 Guardians of Magnesia

The 18 guardians from the *Republic* are shown in the inner circle; those which approximate the chromatic scale are given solid radial lines. The 19 "new arrivals" generated by 7 are shown in the outer circle. If we assume that the "year" begins with a♭ and procede clockwise, then its discrepancy with the square root of 2 falls in the 12th month, assigned to Pluto. The ratios 48:49 and 49:50 are emphasized in the "marriage allegory" of the *Republic*. The trigonometric functions of the tone ratios are shown in chapter 9.

prove to be totally integrated. In obedience to Plato's strictures, I have omitted the radial cut for 3200, reciprocal of 3969, so that ten new arrivals show themselves to us in the left half of the circle, but only nine can be seen in the right half. (I use the symbols (V) and (Λ) for these new "septimal" tones to indicate that they are higher or lower in pitch, respectively, than tones bearing these letters would be in other tunings.) The trigonometric functions discussed in chapter 9 make clear why this circle must be rigidly limited to just 37 elements.

The thirty-seven guardians of Magnesia are listed in figure 37 as a sequence of "superparticular" ratios, 36:35 , etc., the first number being one unit larger than the second, equivalent to the Greek *epimoric, "one* added." This table of ratios shows how unity in the microcosm pervades the unity of Magnesia as a macrocosm. "Citizens of the fourth highest property class," so I presume, are those with prime factors greater than 7.

With the guardians before us it is easier to understand Plato's elaborate emphasis on mathematics in the political theory of *Laws.* In Magnesia proportion rules. *Justice* is defined as "the 'equality' that unequals deserve to get" (757), hence the general method of the rulers must be "to grant much to the great and less to the less great." Magnesians must be educated to understand "the real relationship between commensurables and incommensurables" (820). To be ignorant of such matters is a "swinish stupidity" which makes Plato's Athenian blush, not only for himself "but for Greeks in general" (819). Intervals fathered by the prime numbers 3, 5, and 7 are incommensurable with each other and with the idealized values of $\sqrt{2}$, $\sqrt[3]{2}$, etc., in Brumbaugh's map; Socrates, tongue in cheek, pointed out that the mathematics for his luxurious city (Atlantis) fathered by 3 and 5 would need "swineherds," whereas in his earlier, "essential city" fathered by 3 (Athens) "there was no need" *(Republic* 373c). It can hardly be doubted, I believe, that Plato himself possessed such charts and tables as I have inserted here, and that he used them in his lectures. My own circular diagrams were evolved directly from a procedure hinted at in *Laws:* In Egypt, the Athenian reminds his listeners, "lessons in calculation have been divised for tiny tots to learn while they are enjoying themselves at play." One of their games with "bowls of gold and bronze and silver and so on" can be read as an allusion to our "rotations of the plane" through various angular values determined by 3, 5, 7 and their multiples (819). If Plato's "bowls" were suitably marked, tonal cuts in the octave-circle could have been plotted to any degree of complexity by iterated angular rotations.

Let us turn now to the laws concerning mothers, children, and nursemaids for further insights on Magnesia's guardians.

		1.	D	5040	36:35	(1)	
	49:48 [	2.	d^{v2}	4900			
	64:63 [	3.	$c\#^{\wedge 2}$	4800	28:27	(2)	
50:49 [		4.	c#	4725	Comma	225:224	
		5.	$d\flat^{v2}$	4704	28:27	(3)	
	81:80 [	6.	c	4536			
	64:63 [	7.	C	4480	36:35	(4)	
	126:125 [	8.	c^{v1}	4410	28:27	(5)	
49:48 [	____ [	9.	b#	4375			
		10.	$b^{\wedge 1}$	4320	36:35	(6)	
		11.	b	4200	28:27	(7)	
		12.	$a\#^{\wedge 2}$	4050	Comma	225:224	
	64:63 [	13.	$b\flat$	4032			
	81:80 [	14.	$b\flat^{v2}$	3969	36:35	(8)	
		15.	$b\flat^{v1}$	3920	28:27	(9)	
		16.	A	3780	36:35	(10)	
		17.	a^{v2}	3675	49:48		
	225:224 [	18.	$g\#^{\wedge 2}$	3600			
36:35	64:63 [	19.	$a\flat$	3584	50:49	Pluto	9.8 Interval of Disjunction
	126:125 [	20.	$a\flat^{v2}$	3528			
28:27	____ [	21.	g#	3500	49:48		
	64:63 [	22.	$g^{\wedge 2}$	3456			
	81:80 [	23.	g	3402	36:35	(10)	
		24.	G	3360	28:27	(9)	
		25.	$f\#^{\wedge 1}$	3240	36:35	(8)	
		26.	f#	3150	Comma	225:224	
		27.	$g\flat^{v2}$	3136	28:27	(7)	
		28.	f	3024	36:35	(6)	
	49:48 [	29.	f^{v1}	2940			
	64:63 [	30.	$e^{\wedge 1}$	2880	28:27	(5)	
	81:80 [	31.	E	2835	36:35	(4)	
		32.	e	2800	28:27	(3)	
		33.	$d\#^{\wedge 2}$	2700	Comma	225:224	
	64:63 [	34.	$e\flat$	2688			
	126:125 [	35.	$e\flat^{v2}$	2646	28:27	(2)	
49:48 [	____ [	36.	d#	2625			
		37.	$d^{\wedge 2}$	2592	36:35	(1)	
		1.	D	2520			

FIGURE 37

The Guardians as a Sequence of Superparticular Ratios

MOTHERS, CHILDREN, AND NURSEMAIDS

In Magnesia athletic education begins with the embryo, which must be kept constantly in motion:

> *A pregnant woman should go for walks, and when her child is born she should mould it like wax while it is still supple, and keep it well wrapped up for the first two years of its life…Nurses should persist in carrying the child around until it's three, to keep it from distorting its young limbs by subjecting them to too much pressure (789).*

Since Pythagorean children are *numbers,* and since 3 is the "first" number (1 and 2 being *principles* of number), it is obviously necessary to keep children "well-wrapped up" for the first two years; the first child walks alone (i.e. as an integer cut in the octave-circle) at age 3. And since such children are being prepared to function as reciprocals, mothers and nurses must ensure that children, kept in motion from conception, grow up ambidextrous, both in playing the lyre and in fighting and wrestling (794-795). Magnesia must not suffer the defects of other cities:

> *Almost every state, under present conditions, is only half a state, and develops only half its potentialities, whereas with the same cost and effort, it could double its achievement. Yet what a staggering blunder for a legislator to make (805)!*

Education is compulsory, and the same education is required of both boys and girls. From the ages of three to six both sexes play freely together at the village temples, "kept in order and restrained from bad behaviour by their nurses" (794). The separation of the sexes after age six alludes, I believe, to the fact that in Magnesia the ratios 3:4:5:6 operate freely to produce eighteen guardians from the parent city, numbers $2^p3^q5^r \leq 720$, while the prime number 7 is used with careful restraint to produce nineteen new arrivals, of the form $2^p3^q5^r7^s \leq$ 5,040. Nursemaids themselves are carefully supervised by a group of twelve women. The reason why nursemaids "should be as strong as possible, and there must be plenty of them" (790), can be understood if we divide the numerical values of the thirty-seven guardians by 2^p to let them "stand alone" as male, odd integers of the form $3^q5^r7^s$. This division, carried out in figure 38, not only reveals the "nurses' strength" as the first power of 2 which encloses the integer child, but also calls attention

FIGURE 38

Guardians	*Walking age*	*Nurses' strength*	*Guardians*	*Walking age*	*Nurses' strength*
1. $5040/2^4$	$= 315$	2^9	20. $3528/2^3$	$= 441$	2^9
2. $4900/2^2$	$= 1225$	2^{11}	21. $3500/2^2$	$= 875$	2^{10}
3. $4800/2^6$	$= 75$	2^7	22. $3456/2^7$	$= 27$	2^5
4. 4725	$= 4725$	2^{13}	23. $3402/2$	$= 1701$	2^{11}
5. $4704/2^5$	$= 147$	2^8	24. $3360/2^5$	$= 105$	2^7
6. $4536/2^3$	$= 567$	2^{10}	25. $3240/2^3$	$= 405$	2^9
7. $4480/2^7$	$= 35$	2^6	26. $3150/2$	$= 1575$	2^{11}
8. $4410/2$	$= 2205$	2^{12}	27. $3136/2^6$	$= 49$	2^6
9. 4375	$= 4375$	2^{13}	28. $3024/2^4$	$= 189$	2^8
10. $4320/2^5$	$= 135$	2^8	29. $2940/2^2$	$= 735$	2^{10}
11. $4200/2^3$	$= 525$	2^{10}	30. $2880/2^6$	$= 45$	2^6
12. $4050/2$	$= 2525$	2^{12}	31. 2835	$= 2835$	2^{12}
13. $4032/2^6$	$= 63$	2^6	32. $2800/2^4$	$= 175$	2^8
14. 3969	$= 3969$	2^{12}	33. $2700/2^2$	$= 675$	2^{10}
15. $3920/2^4$	$= 245$	2^8	34. $2688/2^7$	$= 21$	2^5
16. $3780/2^2$	$= 945$	2^{10}	35. $2646/2$	$= 1323$	2^{11}
17. 3675	$= 3675$	2^{12}	36. 2625	$= 2625$	2^{12}
18. $3600/2^4$	$= 225$	2^8	37. $2592/2^5$	$= 81$	2^7
19. $3584/2^9$	$= 7$	2^3			

Guardians as Smallest Integers

Tone values first appear as odd numbers, within a "double defined by some power of 2 which functions as a "nursemaid." Multiplication and division by the female matrix operator "2" never changes the locus of a tone in the tone circle, but the restriction of results to smallest integers means that behind Plato's joke is a serious lesson in number theory. The 37 numbers in figures 36 and 37 are tonally isomorphic with the 37 "walking ages" shown here.

to the vigorous role this "female number" plays in Magnesian arithmetic, justifying the Athenian's insistence that women must surrender all modesty, join the communal meals, and wrestle naked like the men. An uneducated woman, he declares, is an even greater danger to the state than an uneducated man, "perhaps even twice as great" (781).

She is always an even number, twice as great as the male odd number to which she is allied, and defines an octave 2:1, nearly twice as great as the fifth 3:2, largest interval generated by a "man."

THE GUARDIANS AS A TUNING SYSTEM

How Plato might have viewed the musical functions of his guardians is suggested in figure 39 which relates them to various tunings by Archytas, Didymus and Ptolemy. (To simplify the presentation, I have used only the musical notation from figures 36 and 37, which can be consulted for the relevant numbers.) Since guardians far exceed their conceivable tonal functions, I conclude that Plato, having grounded the construction on tonal meanings, actually carries it beyond them for the sake of the astronomers' trigonometry shown in chapter 9. We know little about the Greek modes in his day; further correlations than these would be purely speculative.

MUSICAL POLITICS

Since Plato's Athenian claims that he is trying to follow the Egyptian custom and "sanctify all our dances and music" (799), it is not surprising that the ratios in the scale pervade his legislation. As in all Platonic cities, the marriage legislation carries the highest priority. Magnesia's women must marry between the ages of 16 and 20 4:5); men must marry between 25 and 30 (= 5:6); men not married by 35 (and 30:35 = 6:7, the ratio which generates our nineteen new arrivals) are liable to a fine. The highest class (of largest-property owners) is permitted the greatest expenditures and is similarly required to pay the greatest fines, Magnesia being founded on the notion of "proportional" justice. Allow-ances for wedding feasts for the four classes have the ratios 8:4:2:1 (octaves); for marriage garments 12:9:6:5; for funerals 5:3:2:1. Fines for the four classes for not marrying have the ratios 10:7:6:3; for disrespect of elders 6:5:3:2; for bringing false charges 6:4:3:2; for non-attendance at elections 4:3: 1: 1; for cowardice before the enemy 50: 15:9:3; for failing to confine a madman 25:12:9:6. All of these ratios can be found among the thirty-seven guardians from the various classes, and some may even be infused with a certain amount of humor.

Plato amused himself with *Laws* over the last twenty years of his life, and was in the process of revising it from straight monologue to dialogue at his death, leaving alternate versions of several passages to be-devil translators.

1. Archytas' Diatonic and Ptolemy's Diatonic Toniaion

A		G		f^{v1}		E		D		C		$b^{\flat v1}$		A
	9:8		8:7		28:27		9:8		9:8		8:7		28:27	

2. Didymus' Diatonic

A		G		f		E		D		C		b♭		A
	9:8		10:9		16:15		9:8		9:8		10:9		16:15	

3. Ptolemy's Diatonic Malakon

D		c^{v1}		$b^{\flat v2}$		A
	8:7		10:9		21:20	

4. Ptolemy's Diatonic Syntonon

A		g		f		E		D		c		b♭		A
G		f		e♭		D		C		b♭		a♭		G
f#		E		D		c#		b		A		G		f#
D		c		b♭		A		G		f		e♭		D
b		A		G		f#		e		D		C		b
	10:9		9:8		16:15		9:8		10:9		9:8		16:15	

5. Didymus' Chromatic

A		f#		f		E		D		b		b♭		A
D		b		b♭		A		G		e		e♭		D
	6:5		25:24		16:15		9:8		6:5		25:24		16:15	

6. Ptolemy's Chromatic Malakon

A		f#		f^{v1}		E		D		b		$b^{\flat v1}$		A
	6:5		15:14		28:27		9:8		6:5		15:14		28:27	

7. Archytas' Enharmonic

A		f		f^{v1}		E		D		b♭		$b^{\flat v1}$		A
	5:4		36:35		28:27		9:8		5:4		36:35		28:27	

FIGURE 39

Platonic Guardians as a Tuning System

The Archytas tunings were presumably available to Plato for a model showing how the prime number 7 could be integrated with the prime number 3 which generates perfect fourths and fifths, and with the prime number 5 which generates major and minor thirds. How many of these other tunings recorded by Ptolemy (2nd. c. A.D.) were also known to Plato is uncertain. It is possible that he envisaged his 37 numbers as a "transposition system," but his more serious purpose appears to have been to provide a rigorous introduction to a systematic study of numbers, and in particular of their metric qualities and trigonometric functions.

Laws contains many of his most explicit statements about mathematics, music, education, the soul, politics, and other subjects, yet the dialogue remains not only the most monumental of his works but also the most problematical. I have deliberately ignored all dimensions of the work except the purely musical, the one dimension Plato scholars consistently overlook. Awareness of the purely musical content of *Laws* solves, I believe, most of the numerical puzzles, but it will make life no easier for Platonists who carry the burden of trying to separate Platonic jests from Platonic convictions. Our dramatist is an incorrigible humorist:

> *We set out, by amusing ourselves with laws—it's a dignified game and it suits our time of life (685).*

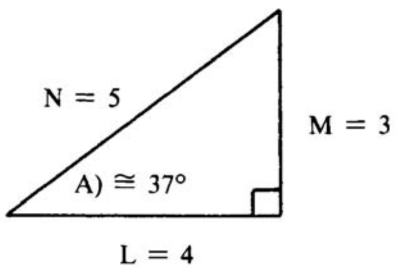
> *These ideas we old men have been tossing about have given us splendid sport (769).*

9
Plato's Musical Trigonometry

The ratios between Plato's 37 guardians of Magnesia and the reference D = 5040, taken in order, define a sequence of right triangles in which acute angles vary successively by an average of 1°. The 37 elements within the hexagonal matrix of the marriage allegory (cf. fig. 4d) perform a similar function. These correlations continue well into the second octave. We thus possess two different sets of Platonic right triangles whose smaller acute angles vary from 1° to 45°, constructed from the ratios of successive tones. I assume that these sets of triangles reveal the tonal foundations of Pythagorean trigonometry, and that they probably motivate, in both the *Republic* and *Laws,* the development of arithmetical material well beyond the practical needs of musicians. Let me first present the data and then explain how it was discovered before speculating on what it means.

Pythagorean triangles (with one right angle and integral sides) are generated by pairs of integers p/q, with p greater than q, according to the following rule:

opposite side	***adjacent side***	***hypotenuse***
$L = 2pq$	$M = p^2 - q^2$	$N = p^2 + q^2$
	Example: $p = 2,\ q = 1$	
$L = 2 \times 2 \times 1 = 4$	$M = 2^2 - 1^2 = 3$	$N = 2^2 + 1^2 = 5$



Different choices of p and q can lead to similar triangles, hence to the same angle. When p/q = 2/1—the all-important octave ratio—they generate a Pythagorean triangle with sides of 3:4:5, containing one angle

FIGURE 40

Trigonometric Functions of the 37 Guardians

	Guardians		*ratio* p/q	$L = 2pq$	$M = p^2-q^2$	$N = p^2 + q^2$	*degrees*
	D	5040	1/1	–	–	–	0
1	d^{v2}	4900	36/35	2520	71	2521	1.61
2	$c\#^{\wedge 2}$	4800	21/20	840	41	841	2.79
3	$c\#$	4725	16/15	480	31	481	3.70
4	$d\flat^{v2}$	4704	15/14	420	29	421	3.95
5	c	4536	10/9	180	19	181	6.03
6	C	4480	9/8	144	17	145	6.73
7	c^{v1}	4410	8/7	112	15	113	7.63
8	$b\#$	4375	144/125	36000	5111	36361	8.08
9	$b^{\wedge 1}$	4320	7/6	84	13	85	8.80
10	b	4200	6/5	60	11	61	10.39
11	$a\#^{\wedge 2}$	4050	56/45	5040	1111	5161	12.43
12	$b\flat$	4032	5/4	40	9	41	12.68
13	$b\flat^{v2}$	3969	80/63	10080	2431	10369	13.56
14	$b\flat^{v1}$	3920	9/7	126	32	130	14.25
15	A	3780	4/3	24	7	25	16.26
16	a^{v2}	3675	48/35	3360	1079	3529	17.80
17	$g\#^{v2}$	3600	7/5	70	24	74	18.92
18	$a\flat$	3584	45/32	2880	1001	3049	19.17
19	$a\flat^{v2}$	3528	10/7	140	51	149	20.02
20	$g\#$	3500	36/25	1800	671	1921	20.44
21	$g^{\wedge 2}$	3456	35/24	1680	649	1801	21.12
22	g	3402	40/27	2160	871	2329	21.96
23	G	3360	3/2	12	5	13	22.62
24	$f\#^{\wedge 1}$	3240	14/9	252	115	277	24.53
25	$f\#$	3150	8/5	80	39	89	25.99
26	$g\flat^{v2}$	3136	45/28	2520	1241	2809	26.22
27	f	3024	5/3	30	16	34	28.07
28	f^{v1}	2940	12/7	168	95	193	29.49
29	$e^{\wedge 1}$	2880	7/4	56	33	65	30.51
30	E	2835	16/9	288	175	337	31.26
31	e	2800	9/5	90	56	106	31.89
32	$d\#^{\wedge 2}$	2700	28/15	840	559	1009	33.64
33	$e\flat$	2688	15/8	240	161	289	33.86
34	$e\flat^{v2}$	2646	40/21	1680	1159	2041	34.60
35	$d\#$	2625	48/25	2400	1679	2929	34.98
36	$d^{\wedge 2}$	2592	35/18	1260	901	1549	35.57
37	D	5040/2	2/1	4	3	5	36.87

FIGURE 40 (continued)

	Guardians		*ratio p/q*	*L = 2pq*	*M = p^2-q^2*	*N = p^2+q^2*	*degrees*
38 = 1	d^{v2}	4900/2	72/35	5040	3959	6409	38.15
39 = 2	$c\#^{\wedge 2}$	4800/2	21/10	420	341	541	39.07
40 = 4	$d\flat^{v2}$	4704/2	15/7	210	176	274	39.97
41 = 5	c	4536/2	20/9	360	319	481	41.54
42 = 6	C	4480/2	9/4	72	65	97	42.08
43 = 7	c^{v1}	4410/2	16/7	224	207	305	42.74
44 = 9	$b^{\wedge 1}$	4320/2	7/3	42	40	58	43.60
45 = 10	b	4200/2	12/5	120	119	169	44.76
46 = 11	$a\#^{\wedge 2}$	4050/2	112/45	10080	10519	14569	46.22
47 = 12	$b\flat$	4032/2	5/2	20	21	29	46.40
48 = 14	$b\flat^{v1}$	3920/2	18/7	252	275	373	47.50
49 = 15	A	4780/2	8/3	48	55	73	48.89
50 = 17	$g\#^{v2}$	3600/2	14/5	140	171	221	50.69
51 = 18	$a\flat$	3584/2	45/16	1440	1769	2281	50.85
52 = 19	$a\flat^{v2}$	3528/2	20/7	280	351	449	51.42
53 = 20	$g\#$	3500/2	72/25	3600	4559	5809	51.70
54 = 21	$g^{\wedge 2}$	3456/2	70/24	3360	4324	5476	52.15
55 = 22	g	3402/2	80/27	4320	5671	7129	52.70
56 = 23	G	3360/2	3/1	6	8	10	53.13
57 = 24	$f\#^{\wedge 1}$	3240/2	28/9	504	703	865	54.36
58 = 25	$f\#$	3150/2	16/5	160	231	281	55.29
59 = 26	$g\flat^{v1}$	3136/2	45/14	1260	1829	2221	55.44
60 = 27	f	3024/2	10/3	60	91	109	56.60
61 = 28	f^{v1}	2940/2	24/7	336	527	625	57.48
62 = 29	$e^{\wedge 1}$	2880/2	7/2	28	45	53	58.11
63 = 31	e	2800/2	18/5	180	299	349	58.95
64 = 32	$d\#^{\wedge 2}$	2700/2	56/15	1680	2911	3361	60.01
65 = 33	$e\flat$	2688/2	15/4	120	209	241	60.14
66 = 34	$e\flat^{v2}$	2646/2	80/21	3360	5959	6841	60.58
67 = 36	$d^{\wedge 2}$	2592/2	35/9	630	1144	1306	61.16
68 = 37	D	5040/4	4/1	8	15	17	61.93

of approximately 37°. If musical guardians are expected to perform the trigonometric function of generating Pythagorean triangles varying by successive degrees, then there must be exactly 37 of them to the octave. In chapter 8 we could find musical employment for only 18 of these guardians (cf. fig. 39), exploring all of the tunings Ptolemy recorded. Plato's apparently superfluous musical guardians, essential only to trigonometry, thus reveal the breadth of concerns he is trying to integrate.

The 37 guardians of Magnesia are displayed in figure 40 with both tonal and trigonometrical functions indicated. The trigonometrical ratios

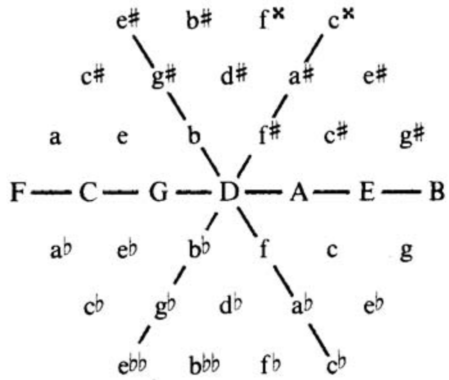
FIGURE 41

Symmetric Trigonometric Functions within 432,000

		tones	ratio p/q	L = 2pq	M = $p^2 - q^2$	N = $p^2 + q^2$	degrees
	D	432,000	1/1	—	—	—	0
1	c×	421,875	128/125	32000	759	32009	1.36
2	d♭	414,720	25/24	1200	49	1201	2.34
3	c♯	405,000	16/15	480	31	481	3.70
4	c♯	400,000	27/25	1350	104	1354	4.41
5	c	388,800	10/9	180	19	181	6.03
6	C	384,000	9/8	144	17	145	6.73
7	b♯	375,000	144/125	36000	5111	36361	8.08
8	c♭	373,248	125/108	27000	3961	27289	8.35
9	c♭	368,640	75/64	9600	1529	9721	9.05
10	B	364,500	32/27	1728	295	1753	9.69
11	b	360,000	6/5	60	11	61	10.39
12	b♭	345,600	5/4	40	9	41	12.68
13	a♯	337,500	32/25	1600	399	1649	14.–
14	b♭♭	331,776	125/96	24000	6409	24841	14.95
15	A	324,000	4/3	24	7	25	16.26
16	a	320,000	27/20	1080	329	1129	16.94
17	a♭	311,040	25/18	900	301	949	18.49
18	a♭	307,200	45/32	2880	1001	3049	19.17
19	g♯	303,750	64/45	5760	2071	6121	19.78
20	g♯	300,000	36/25	1800	671	1921	20.44
21	g	291,600	40/27	2160	871	2329	21.96
22	G	288,000	3/2	12	5	13	22.62
23	f×	281,250	192/125	48000	21239	52489	23.87
24	g♭	276,480	25/16	800	369	881	24.76
25	f♯	270,000	8/5	80	39	89	25.99
26	f	259,200	5/3	30	16	34	28.07
27	F	256,000	27/16	864	473	985	28.70
28	e♯	253,125	128/75	19200	10759	22009	29.26
29	e♯	250,000	216/125	54000	31031	62281	29.88
30	f♭	248,832	125/72	18000	10441	20809	30.12
31	E	243,000	16/9	288	175	337	31.26
32	e	240,000	9/5	90	56	106	31.89
33	e♭	233,280	50/27	2700	1771	3229	33.26
34	e♭	230,400	15/8	240	161	289	33.86
35	d♯	225,000	48/25	2400	1679	2929	34.98
36	e♭♭	221,184	125/64	16000	11529	19721	35.78
37	D	216,000	2/1	4	3	5	36.87

FIGURE 41 (Continued)

		tones	*ratio p/q*	*L = 2pq*	*M = $p^2 - q^2$*	*N = $p^2 + q^2$*	*degrees*
38 = 2	*d♭*	*207,360*	*25/12*	*600*	*481*	*769*	*38.72*
39 = 3	*c♯*	*202,500*	*32/15*	*960*	*799*	*1249*	*39.77*
40 = 4	*c♯*	*200,000*	*54/25*	*2700*	*2291*	*3541*	*40.32*
41 = 5	*c*	*194,400*	*20/9*	*360*	*319*	*481*	*41.54*
42 = 6	*C*	*192,000*	*9/4*	*72*	*65*	*97*	*42.08*
43 = 7	*b♯*	*187,500*	*288/125*	*72000*	*67319*	*98569*	*43.08*
44 = 8	*c♭*	*186,624*	*125/54*	*13500*	*12709*	*18541*	*43.27*
45 = 9	*c♭*	*184,320*	*75/32*	*4800*	*4601*	*6649*	*43.79*
46 = 10	*B*	*182,250*	*64/27*	*3456*	*3367*	*4825*	*44.25*
47 = 11	*b*	*180,000*	*12/5*	*120*	*119*	*169*	*44.76*



p/q, which necessitate exactly 37 tones to the octave, are derived from figures 36 and 37 in chapter 8. Degrees can be computed from the ratios of the three sides of a triangle according to any modern trigonometrical table.[1] Notice that the *order* number of the guardians corresponds approximately to the degrees up to 52° before the correlation begins to fail. Since Plato requires his trainees to travel twice around the circle of Magnesia (a full octave each way), I have charted the ratios through two full octaves although the useful ratios stop at b = 12/5 corresponding approximately to 45°, since complementary angles carry the correlation to 90°.

The 37 elements within the hexagonal matrix of the marriage allegory are similarly arrayed in figure 41, where I have stopped at the 47th triangle, D: b = 12/5, corresponding approximately to 45°. These two sets of triangles share 18 elements (the 18 guardians from the parent city, in the calendrical circle of figure 34). Correlations between successive tones and successive triangles are excellent through about 42°; two of the last four elements in this set ought to be eliminated to make the 45th triangle match 45°. The 37 elements in this set require as common denominator the Hindu *Kali Yuga* number 432,000 (= 2 × 60^3), a characteristically Babylonian number prominent in the mythology of the ancient world. It can be inferred that the 37 guardians of *Laws* were conceived as Platonic competitors for the inherited hexagonal "net" of a far more ancient set of 18 pairs of twins surrounding their guardian deity (cf. figs 4d, 8 and 31).

Plato's new set of 37 elements in *Laws* is compared through 45° with the older set from the *Republic* in the graphs of figure 42. Values seldom coincide, but both sets deviate in similar ways around the norm of 1° per guardian. There are no superfluous elements in the *Laws* set as against two in that from the *Republic.* (Note that elements 3 and 8 in *Laws* cannot be halved for the second octave.) A comparison of the numbers in figures 40 and 41 shows that *Laws,* which needs a least common denominator of only 5,040, also uses somewhat fewer large numbers in the triangle computation than the *Republic,* whose set requires a least common denominator of 432,000. We are witnessing, apparently, an experimental confrontation between a Greek decimal construction using four primes (2, 3, 5 and 7) and an older Babylonian sexagesimal one using only the first three. Ptolemy's spectacular achievements, using Babylonian sexagesimal arithmetic, doomed Plato's effort (if, indeed, it was ever taken very seriously). The rather devious selection of 37 elements in *Laws* suffers something by comparison with the straightforward selection of 37 in the hexagonal matrix of the marriage allegory, which consists of all the "twins" available on a monochord-diameter of 432,000 units, (i.e., every integer q which can be paired with a tonal reciprocal q' —proven by its ability to function in the proportion 60^3:q::q':2 × 60^3—automatically acquires a trigonometric function also, p always representing 2 x60^3 = 432,000 after ratios are reduced to numbers which are relatively prime.)

The prototype for these Platonic constructions can be found on an ancient Babylonian cuneiform tablet known as Plimpton 322, dating from the period 1900-1600 B.C.[2] The tablet, which has enjoyed an extensive commentary by mathematicians, contains the formulas for 15 Pythagorean triangles varying by approximately 1° in the range from 45° to 31°. The Plimpton numbers, in decimal form, are summarized in figure 43, together with their tonal correlations and a graph of their trigonometrical success in approximating a variation by 1°.

The p/q numbers for the Plimpton triangles fall within the perfect fourth 4:3 between $^{12}/_5$ and $^9/_5$. As soon as that was recognized, it seemed

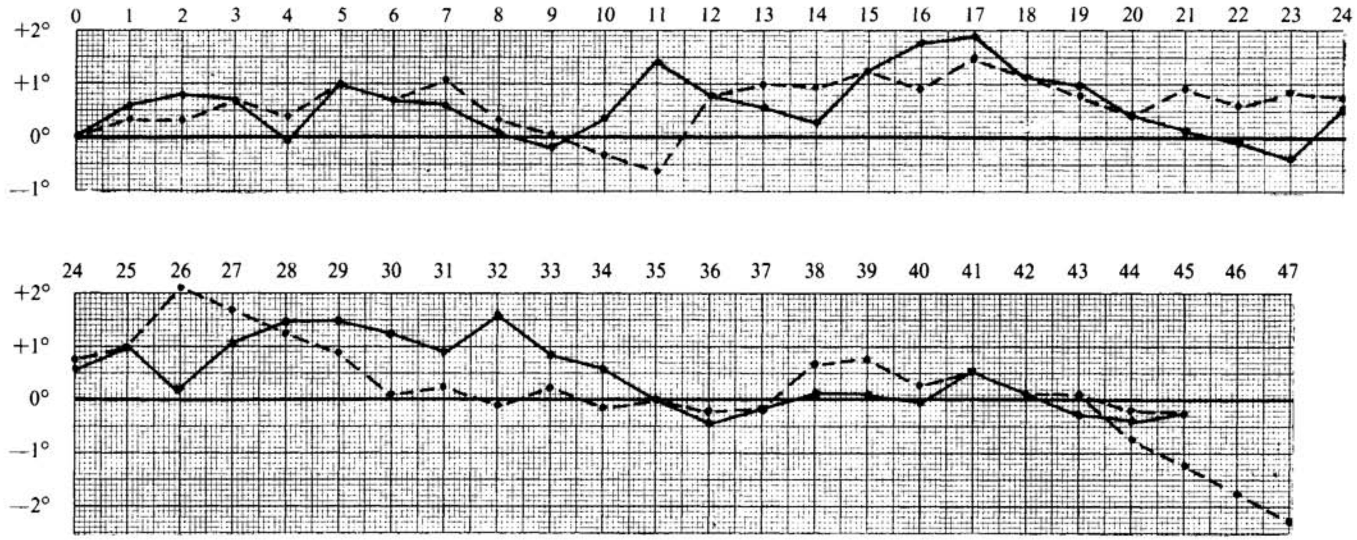


FIGURE 42

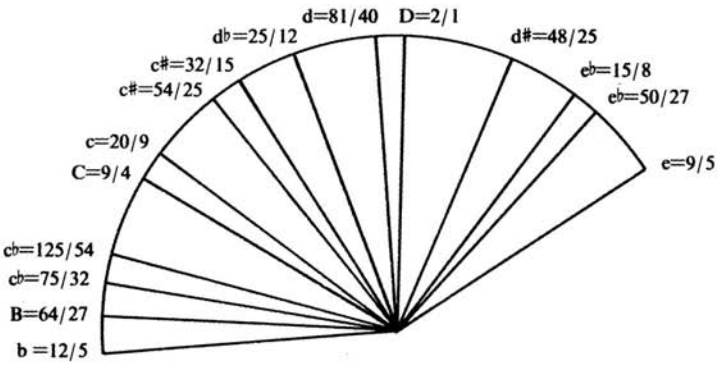
Consecutive Guardians as Generators of Consecutive Pythagorean Triangles

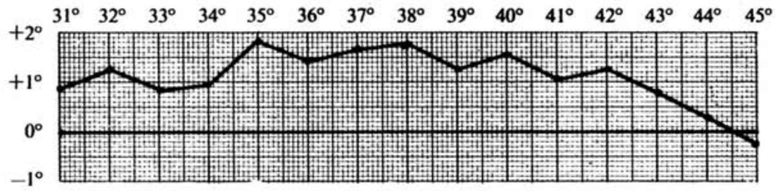
The horizontal axis of the graph correlates consecutive guardians with consecutive degrees in the Pythagorean triangles they generate. The vertical dimension is intended to exaggerate discrepancies between the Archytas tuning of *Laws* (________) and the Just tuning of the *Republic* (– – – –). The flatter the curve the more accurately successive ratios correlate with successive degrees. The *Republic* curve can be improved by the deletion of elements 44 and 46.

FIGURE 43

Tonal Interpretation of Plimpton 322

Tone	*line*	*ratio*	*2pq*	$p^2 - q^2$	$p^2 + q^2$	*degrees*
e	*17*	*9/5*	*90*	*56*	*106*	*31.89*†*
e♭	*16*	*50/27*	*2700*	*1771*	*3229*	*33.26**
e♭	*15*	*15/8*	*240*	*161*	*289*	*33.86*†*
d#	*14*	*48/25*	*2400*	*1679*	*2929*	*34.98**
D	*13*	*2/1*	*60*	*45*	*75*	*36.87*†*
d	*12*	*81/40*	*6480*	*4961*	*8161*	*37.44*
d♭	*11*	*25/12*	*600*	*481*	*769*	*38.72**
c#	*10*	*32/15*	*960*	*799*	*1249*	*39.77*†*
c#	*9*	*54/25*	*2700*	*2291*	*3541*	*40.32**
c	*8*	*20/9*	*360*	*319*	*481*	*41.54*†*
C	*7*	*9/4*	*72*	*65*	*97*	*42.08*†*
c♭	*6*	*125/54*	*13500*	*12709*	*18541*	*43.27**
c♭	*5*	*75/32*	*4800*	*4601*	*6649*	*43.79**
B	*4*	*64/27*	*3456*	*3367*	*4825*	*44.25**
b	*3*	*12/5*	*120*	*119*	*169*	*44.76*†*





*Also in figure 41 *(Republic)*

†Also in figure 40 *(Laws)*

possible that the succession of commas, quartertones, and other small tonal intervals between them might explain the similar crowding of tones among the 37 guardians of Magnesia. A check of the triangles the 37 tone-ratios generated disclosed that they did indeed vary by an average of 1°. The simple fact that the octave $2/1$ generates a triangle with an angle of 37°—thus determining in advance the general nature of such a correlation—was understood, only in retrospect. The fact that the hexagonal matrix of the marriage allegory also

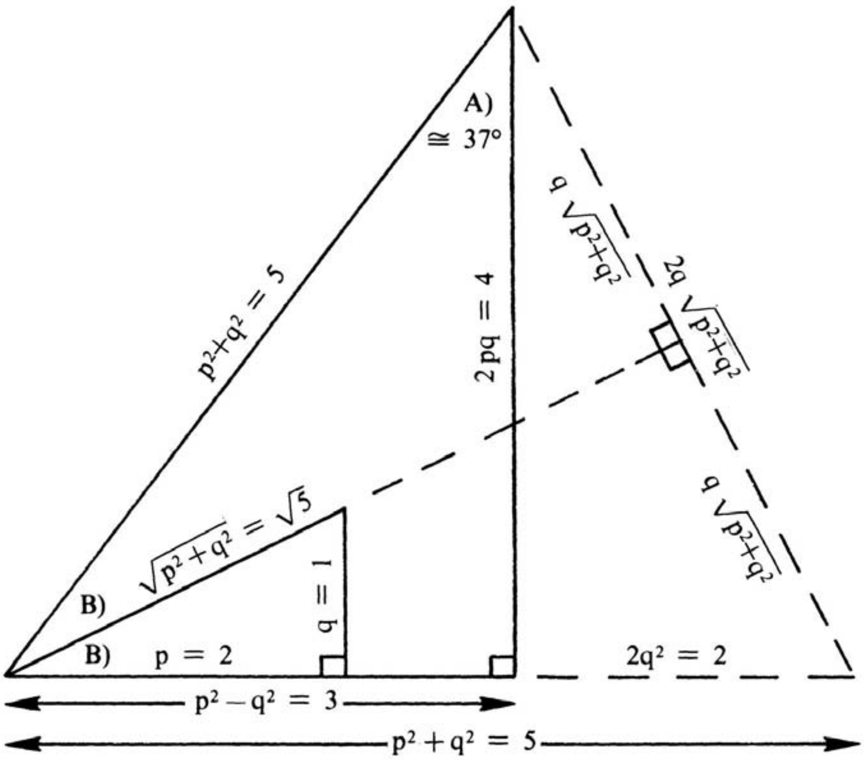


FIGURE 44

The Construction of Pythagorean Triangles

In every Pythagorean triangle, one of the acute angles can be expressed as the sum of two equal angles according to the following formula:

$$\text{tangent B} = \frac{q}{p} \qquad \text{tangent 2B} = \frac{2pq}{p^2 - q^2}$$

The angular values given in this chapter are not the summation angles. however, but the resultant complementary angles A.

happens to contain 37 elements made it imperative to check their triangles against those of *Laws,* disclosing that their generally simpler process of selection actually gives comparable, and in many cases somewhat superior, trigonometric results.

There is a further level of musical implication in these triangular numbers. The relatively prime tone numbers can be taken directly as the two sides of a right triangle; in many cases, however, the hypotenuse would prove irrational (for instance, sides of 2:1 produce an hypotenuse of $\sqrt{2^2+1^2} = \sqrt{5}$. Two such triangles, however, can always be added geometrically to produce a Pythagorean triangle.[3] The ratio $^2\!/_1$ (item 37 in both the *Laws* set and in the *Republic* set) generates the Pythagorean triple 3, 4, and 5; in figure 44 we see how this larger triangle can result from the geometric doubling of the original angle determined by p/q = $^2\!/_1$. In short, the tone ratios can be used geometrically in either of two ways.

One wonders whether older cultures which knew the musical scale and a method for adding triangles geometrically needed the abstract algebraic concepts for generating Pythagorean triples. The conception of a circle divided into 360 degrees, now attributed to the time of Hypsicles (2nd c. B.C.), seems implicit in Plato's constructions. Could older cultures have been clearer about “degrees” than we have thought, and as Plimpton 322 hints?[4] Or did they possess the hexagonal matrix among their sacred symbols without recognizing its trigonometrical potential? These are questions not for musicians but for historians of science to answer. Our Pythagorean Plato, however, is unmasked as a man delighting in the mathematical advantages of a musical education.

10
Conclusions

"All that is beautiful is difficult," Plato wrote in the last words of *Greater Hippias*. His own works remain as difficult as they are beautiful. Even if the musical interpretations of the *Republic, Timaeus, Critias, Statesman* and *Laws* offered here should be found acceptable as "the likely story" of what Plato meant by his continuous thread of "muses' jests," all growing out of the marriage allegory of the *Republic,* our musical solutions to these mathematical riddles provoke new questions. What is needed is a new dialogue, and one from which musicians are not excluded. It is with the hope of encouraging a new interdisciplinary dialogue that I allow myself a few tentative conclusions here concerning Plato's thoughts on music, mathematics, astronomy, political science, and philosophy in general.

MUSIC

Plato clearly conceived "equal temperament." His discourse on Just tuning in the marriage allegory, on Pythagorean tuning in the tyrant's allegory, and on the tempering of the Sirens' scale in the myth of Er covers all of the interesting aspects of the problem and describes its solution. In *The Myth of Invariance,* I developed the reasons for believing that his basic material was the common property of the major middle-eastern civilizations for the preceding three thousand years, but he must be credited as being the oldest extant source of an explicit treatise on this subject. His inspiration for equal temperament may have been partly a somewhat mistaken theory of planetary motion.

The best known and least understood Platonic opinion on music concerns his limitation, in ideal cities, to the Dorian and Phrygian modes. Here, in full, is Socrates' discussion with Glaucon in the *Republic:*

> *"We said there is no further need of wailing and lamentations in speeches."*
>
> *"No, there isn't."*

"What are the wailing modes? Tell me, for you're musical."

"The mixed Lydian," he said, "and the 'tight' Lydian and some similar ones."

"Aren't they to be excluded?" I said. "They're useless for women who are to be decent, let alone for men."

"Certainly."

"Then again, drunkenness, softness, and idleness are most unseemly for guardians."

"Of course."

"What modes are soft and suitable for symposia."

"There are some Ionian," he said, "and some Lydian, too, which are called "slack.' "

"Could you, my friend, use them for war-making men?"

"Not at all," he said. "So, you've probably got the Dorian and the Phrygian left."

"I don't know the modes," I said. "Just leave that mode which would appropriately imitate the sounds and accents of a man who is courageous in warlike deeds and every violent work, and who in failure or when going to face wounds or death or falling into some other disaster, in the face of all these things stands up firmly and patiently against chance. And, again, leave another mode for a man who performs a peaceful deed, one that is not violent but voluntary, either persuading someone of something and making a request—whether a god by prayer or a human being by instruction and exhortation—or, on the contrary, holding himself in check for someone else who makes a request or instructs him or persuades him to change, and as a result acting intelligently, not behaving arrogantly, but in all these things acting moderately and in measure and being content with the consequences. These two modes—a violent one and a voluntary one, which will produce the finest imitation of the sounds of unfortunate and fortunate, moderate and courageous men—leave these."

"You're asking me to leave none other than those I was just speaking of (398-399c)."

Since the Greek Phrygian mode is the only one which retains the same pattern of tones and semitones under reciprocation, a central Platonic concern, and since the Dorian mode together with its reciprocal define all eleven of the tones which Pythagorean symmetry and rational numbers can establish withoutinternal conflict, I assume that Plato excludes all others simply on the ground that they were not "necessary" his most important criterion for what belongs in an ideal city *(Republic* 369d, 373a). His comments on "the mixed

Lydian" and "the 'tight' Lydian," whatever they were, as "wailing modes," and on the Ionian and "slack" Lydian as "encouraging drunkenness, softness and idleness," were probably recognized by his audience as typical jokes, based on the fact that certain strings were "tight for some modes and "slack" for others. Aristides Quintilianus (3rd or 4th c. A. D.) recorded tunings for six modes which may or may not be those of Damon, the fifth-century B.C. musician whom Plato was fond of quoting. Although it is impossible to verify these tunings, I list them in figure 50 of the historical appendix to show that the Dorian tones include all of the others and that the guardians of Magnesia provide for them. It seems obvious that Plato has given posterity more credit than he ought for a sense of humor and a musical understanding. The last two pages of Aristotle's *Politics* set the record straight, lauding Dorian as "manliest" of the modes, but giving each of the others loving commendation.

MATHEMATICS

Plato's mathematical allegories focus on the problem of divding a cyclic octave 1:2 into twelve equal parts, a task which can be achieved only by introducing the irrational $\sqrt[12]{2}$. Since all the various "Pythagorean" tunings, whatever their ratios, can merely approximate such equal divisions of the continuum of real number, Platonic tuning theory—viewed mathematically—is part of what today is called Diophantine approximation. Historians of mathematics have possessed hitherto a very considerable body of theoretical materials from Plato's century, culminating in Euclid. Plato's mathematical allegories, when eventually purified from any contaminating ideas I may have introduced inadvertently, should add something to our understanding of computation in his century.

It is interesting that Plato's constructions exhibit so many similarities with our modem notion of a formal mathematical group. His awareness of the "associative" and "commutative" properties is indicated by the uncertainty of paternity in his cities. He is obviously interested in "identity" elements and adamant that reciprocals ("inverses" for each element) be studied. His awareness of the "closure" principle is exhibited in his passion for interfering with its operation, rigidly limiting populations whose "marriage laws" would otherwise generate infinite groups. The formal group properties are implicit in his formulas, yet by limiting "form numbers" (meaning "group generators") to the first ten integers, Plato invites the implications that his own examples should not be generalized to allow "form numbers" to exceed 10 and that his "rotations of the plane" (i.e. his tone-circles) should not include other rotations than those he himself used. For Plato to have approved so generous a mathematical attitude would have divorced his number theory from its limited ground of musical applicability, and would have dissolved the principle of "limitation," and particularly "self-limitation," which lies at the heart of his political and musical theories. Aristotle sundered these unities.

ASTRONOMY

Plato wrote no treatise on astronomy, although Platonists who cannot read the musical implications of the myth of Er or of the *Timaeus* World-Soul have supposed otherwise. He studied astronomy "by the use of problems, as in geometry," and many of his problems were musical ones.

POLITICAL THEORY

Plato's political theory is as rigorously musicalized as his astronomy. His legislation on men, women, children, armies, elections, governing bodies, fines, fees, and a host of other problems is grounded, in each political dialogue, on his relevant musical-mathematical model. Beneath every noble sentiment lurks a mathematical jest. Yet his abstract models, despite his humor, remain very beautiful examples of "communities" with varying degrees of internal complexity and various solutions to the problem of limitation. Plato the writer, however, has fused the sublime and the ridiculous so perfectly that we are not likely ever to separate them successfully.

Harvey Wheeler has defined political theory as "the symbolic idiom which is concerned with cultural institutions as manipulable objects."[1] From that point of view, political theory begins with the *Republic,* which "projected a total culture into the heavenly realm of pure form... the first explicitly theoretical approach to cultural forms." Concerning Plato's relation to Aristotelian political theory, Wheeler notes that:

> *The aesthetic prefiguration came first... After the Platonic objectification of culture from aesthetic models Aristotle way able to*

> *perform a further logical operation and shunt the aesthetic models and visualize the functional models.*

Plato's Pythagoreanism can be understood in Wheeler's terms as the necessary “aesthetic prefiguration” for political theory.

PHILOSOPHY

What kind of a philosopher is our “Pythagorean Plato”? He is, I believe, what Antonio T. de Nicolás has called a four-dimensional man.”[2] De Nicolás coined that phrase to describe a radical philosophical activity he discovered was implied among the poets of the Ṛg Veda, India's oldest sacred book. It is the activity of philosopher-poets for whom music “embodies” the secrets of the universe. For Plato, as for his Hindu predecessors, *sound* was the primary guide to “interiority.” *Movement* is the “embodied movement” of the soul through which creation becomes manifest. Without ordered movement we are in the field of “non-being.” Music, being an art of pure relations, offers the primary examples of aesthetic “being.” Despite Plato's emphasis on vision as the most important of the senses, he is actually directing attention to visual models of sound phenomena while asking us to rise above this ground of appearances and contemplate with him the invariance of pattern.

De Nicolás' studies into Hindu culture have made it quite clear that there are two different and contradictory models grounding the meaning of sentences, cultures, and whole philosophies. One model takes sight and its criteria as the primary organizer of sensation. The other takes sound and its criteria as the primary organizer of sensation. On the model of sight a language of substances is born to communicate exactly what the model had previously established: atomic things and events, within a visual space ruled by fixed coordinates of space and time. On the model of sound, on the other hand, a language is born for communication which emphasizes *perspectives* not of the same fixed object but of a multitude of relations which must appear for any one object seemingly to appear. De Nicolás suggests that the relation between the language of substance and the language of perspectives is the relation of complementarity as understood by modern Physics: what can truly be said in one language cannot truly be said in the other, and vice versa.[3] Whatever the case, it is always a challenge to the philosopher to discover the extent to which he himself is the victim of the model through which he tries to infuse meaning into the world. And this applies as much to Plato, Aristotle, or the present writer.

De Nicolás has schematized Hindu thought in a way which applies very beautifully to Plato, and to others similarly oriented by aesthetic criteria. The "four dimensions," or "intentionalities," or "languages" de Nicolás employs are those of 1) Non-Existence ("non-being" in Greece), 2) Existence ("being"), 3) Images and Sacrifice, and 4) Embodied Vision.

1) Plato's concern with "Non-being" is apparent in the confusion of Sameness and Difference which precedes the creation of the World-Soul in *Timaeus,* and in the confusion of values which Socrates forces his friends to concede at the beginning of each exercise in dialectics, and also in his postulate of the inevitable dissolution of all structures. however nobly conceived, into "unlikeness and inharmonious irregularity" with the passage of time. It is in the nature of all the arts which exist in time that their phenomena must continually fade into the ground or chaos of non-being from which they have emerged. Socrates is wedded to this aesthetic ground when he denies over and over, despite the protestations of his friends, that he really *knows* anything. His own convictions are subject to time and circumstance, hence it is not modesty but wisdom which restrains him.

2) Plato's concern with "Being" is exemplified, on the one hand, by his search for definitions and, on the other, by his use of mathematical-musical models which are explicit. His insistence that "true-being" belongs only to the idealized patterns of things may appeal more to musicians than to others with a different philosophical perspective, but his favorite examples are topological sets belonging to the domain of pure mathematics as well as to the applied science of acoustical theory. His emphasis on "logical realism," as it has been called, gave a powerful thrust to the development of pure mathematics in an age which still was immersed in the ancient cosmology with its essentially poetical, and less-stringently rational, outlook.

3) Plato's concern with "Images and Sacrifice" can be understood musically from the four different tuning systems which model his four communities, each vividly presented, but each necessarily "sacrificed" to let the alternatives come into view. This mode of thinking, or manner of talking, is appropriate for the realm of *alternative* aesthetic structures which are equally appealing—from one point of view or another—but mutually incompatible in time. Not only do Plato's musical cities come to be, but each must pass away—as each tone, each mode, each rhythm must pass—to allow the next to come into focus. There is no dialogue which "fixes" Plato's thought for us. The *Republic* and *Laws* are so opposed in spirit—the first proposing a communal brotherhood with few laws and common wives, children and property, the

second satiated with law and the rigid authority of central government—as to appear to be the work of two different men. His other dialogues show a similar readiness to "sacrifice" a perspective laboriously established in an earlier one. In short, Plato is in no sense what we have come to understand as a "Platonist." Neither is he a Pythagorean. His Pythagoreanism is but a *prelude* to philosophy, to "the song itself ... which dialectic performs," a prelude to which Platonists have declined to listen, although it establishes the multiple perspectives from which Plato understood himself.

4) What Plato possessed, and what he tried to inculcate, was what de Nicolás has termed "Embodied Vision." Such a vision is grounded in the arts and sciences of one's culture when they are treasured in rich enough measure that one possesses *alternative* perspectives on any phenomenon. Although Plato "has a reputation for deprecating arts and crafts, and for short-changing physical reality and empirical encounters with it," Robert Brumbaugh points out that Plato himself actually does neither: "Plato's own references to the arts and crafts... are ubiquitous and encyclopedic."[4,5] The art of mathematics holds a very special place in Plato's esteem: it is essentially an art of changing viewpoints, of alternative perspectives, and it gives to those who pursue it seriously a release from the imprisonment of a single viewpoint—a real freedom of thought—characteristically Vedic as well as Platonic.

Alfred North Whitehead summarized the history of Western philosophy as essentially a footnote to Plato. But Plato himself, de Nicolás observes, is but a kind of footnote to even more ancient philosophers. With the rise of Aristotelian rationalism and the later emergence of theological dogmatism, Plato's mode of thinking was misunderstood and misrepresented, and with fateful consequences. Plato himself invalidates all efforts to ground dogmatic philosophy on his authority. Thanks to the survival of the whole corpus of his writings, Plato remains the only truly valid or even essential—commentator on himself. The vision of the world which he embodied—as "a visible living creature...a perceptible god" is a vision we are trying to recapture.

Appendix I
Historical Commentary

Only after completing my analysis of Plato's mathematical allegories did I begin to realize that he himself is the best guide to their meaning. I have tried to let him tell his own story here with as few interruptions as possible. My own guides, however, included not only the modern scholars I have mentioned along the way but also Aristotle, Aristoxenus, Speusippus, Xenocrates, Crantor, Euclid, Ptolemy, Plutarch, Nicomachus, Aristides, Proclus, and Boethius. This study would be incomplete if I failed to, make clear my debts to each of them. But they present an enormous problem: not one of these men has ever been studied from the perspective provided by Plato' s Pythagoreanism. We have read all of them “out of context.” My purpose here is to point out some of the problems which future studies must explore.

ARISTOTLE

Next to Plato himself, the most helpful guide to Plato's Pythagorean tone-models is Aristotle, who eventually detested them. The writings of Aristotle provide two kinds of useful information: direct explanations, usually brief, often mere asides within some general argument; and scornful mockery of the literal implications of Platonic metaphor, together with explicit criticism of Plato's procedure. When Aristotle is re-studied from our new musical perspective, we shall probably gain renewed admiration for both men; for Plato's Pythagoreanism is clearly part of the matrix within which Aristotle forges the logical tools for the direct study of nature itself, as opposed to Plato's analogical procedure.

Aristotle, like Plato, is fond of musical analogies for philosophical problems, confirming the general musicality of ancient Athens, but it is his specific information for which we can be grateful. Significantly, the ratio 2:1 is archetype (or form, or essence) of the musical octave *(Physics* 194b).[1] The scale consists of seven strings *(Metaphysics* 1093a), and “the middle string is the beginning” (ibid., 1053a and 1018b). (I have charted *Mese,* “middle,” as D throughout this study, so that octaves and tetrachords both begin on it.)

There are two quartertones—i.e., 27:28 and 35:36 in the Archytas tuning of chapter 8— "not to the ear, but as determined by the ratios" *(Metaphysics* 1053a). There are two basic modes, Dorian and Phrygian; "the other arrangements of the scale are comprehended under one or other of these two" *(Politics* 1290a). The tuning of these modes is similar, distinctions resting on tonal *functions:* "A scale containing the same sounds is said to be different, accordingly as the Dorian or the Phrygian mode is employed" *(Politics* 1276b). These few comments contain more technical information about music than can be found in the whole corpus of Platonic dialogues. "They say," Aristotle writes, that the twenty-four letters from alpha to omega are equal in distance "from the lowest note of the flute to the highest," and the number of this highest note "is equal to that of the whole choir of heaven" *(Metaphysics* 1093b).[2]

Aristotle is equally explicit about Pythagorean attitudes toward numbers and about Plato's use of numbers for ideal Forms: Pythagoreans "supposed the elements of numbers to be the elements of all things, and the whole heaven to be a musical scale and a number" *(Metaphysics* 986a). Pythagoreans say that "the world and all that is in it is determined by the number three" *(On the Heavens* 268a). "They try to work on the assumption that the series of numbers up to 10 is a complete series" *(Metaphysics* 1084a). Pythagoreans "defined justice without qualification as reciprocity" *(Nicomachean Ethics* 1132b). Pythagoreans "identify the infinite with the even"; Plato, however, "has two infinites, the Great and the Small," which he does not actually use because "the infinite in the direction of reduction is not present, as the monad (1) is the smallest . . ." *(Physics* 206b). We remember that Socratic fractions—which would lead to the infinitely small—are always disguised behind least common denominators. Plato "expressly identified" numbers with "the Forms or principles" *(On the Soul* 404b). His "Great and Small" constitute the "matter" of *Timaeus (Physics* 210a). One further comment on proportion may well prove Aristotles' most significant contribution to our understanding of the *Republic* as an essay in temperament: "A single proportion . .. may be relaxed and yet persist up to a point" *(Nicomachean Ethics* 1173a). A "relaxed proportion" which can "persist up to a point" describes what Levarie and Levy mean by a ratio treated as a "norm,"—" at the least a reference point, at the best an ideal toward which one strives," a model for a citizen who does not always demand "exactly what he is owed."

Aristotle's criticism of Platonic-Pythagorean mathematical habits constitutes indirect support for the musical interpretations presented here: "It is paradoxical that there should be an Idea of 10, but not of 11, nor of numbers" *(Metaphysics* 1084a). "To connect the infinite with the all and the whole is not like joining two pieces of string; for it is from this they get the dignity they ascribe to the infinite" *(Physics* 207a). "It is not the numbers that are the essence or the cause of the form; for the ratio is the essence, while the number is the matter" *(Metaphysics 1092b).*

Aristotle's criticism of Plato's political theory offers similarly indirect support for our interpretations: "The error of Socrates must be attributed to the false notion of unity from which he starts" *(Politics* 1263b). It is the nature of a state to be a plurality, Aristotle observes; if we acted on Socrates' assumption that "the greater the unity of the state the better" *(Politics* 1261a), we should find it becoming "an inferior state, like harmony passing into unison, or rhythm which has been reduced to a single foot" *(Politics* 1263b). As for the communal ownership of wives and children, which the shared boundaries of the octave and the alternate routes between the tones force on Socratic political theory, Aristotle is caustic: "How much better is it to be the real cousin of somebody than to be a son after Plato's fashion" *(Politics* 1262a)! One further remark is vicious:

> *How strange it is that Socrates, after having made the children common, should ... permit love and familiarities between father and son or between brother and brother, than which nothing can be more unseemly.*
>
> *(Politics* 1262a)

It is true that the rules of the *Republic* imply approval of such relations, but Aristotle knows very well that such sentiments are not Plato's, and that it is unfair to abuse a man by quoting him out of context. (In this case, male odd numbers must multiply each other to produce further male "children.") Aristotle's "disloyalty" has occasioned considerable comment.

The reasons for it, I believe, are twofold. One reason is that fellow Athenians needed protection from their own simplistic tendencies; we know from history that Platonic metaphor was taken literally and therefore misread by students less astute than Aristotle. The other reason Aristotle himself states in various ways:

> *Mathematics has come to be identical with philosophy for modern thinkers... The whole study of nature has been annihilated.*
>
> *(Metaphysics* 992)

In framing an ideal we may assume what we wish, but should avoid impossibilities.

(Politics 1265a)

Those who use the language of proof we must cross-examine.

(Metaphysics 1000a)

Today not even Socrates' best friends credit him with being entirely fair in an argument; Aristotle was but giving him a taste of his own medicine, with thrusts at every weakness. We need not feel sorry for Plato; he evoked this brilliance.

We owe to Aristotle the birth of musicology as a subject in its own right, with the ear—not number—as the arbiter of events. His own *Politics* closes with an extensive dissertation on the musical modes, of central importance to the *Republic,* rectifying the distortions Socrates introduced for schematic reasons. The dialectical relationship between the founder of the Academy and his seemingly disloyal student remains one of the puzzles in Western philosophy.[3] I believe we must accept Plato's Pythagoreanism in order to make sense of Aristotle, and Aristotle in turn will someday help us understand just how Plato is to be distinguished from the Pythagoreans whose material he used with such virtuosity. Aristotle tried to be fair:

The discourses of Socrates are never commonplace; they always exhibit grace and originality and thought; but perfection in everything can hardly be expected.

(Politics 1265a)

SPEUSIPPUS AND XENOCRATES

Upon Plato's death in 347 B.C., Speusippus, his nephew, became master of the Academy, and upon his death in 339 B.C., Xenocrates, Aristotle's friend, was elected to that position, serving for the next twenty-five years. It is interesting that neither Speusippus, Xenocrates, nor Aristotle could accept Plato's views on the relations between forms and numbers. According to Aristotle, who chronicles their disputes in books XIII and XIV of his *Metaphysics,* they held four distinct positions:

Plato: Two kinds of number exist. The first ten integers are form numbers, identical with “ideas” and distinct from “mathematical numbers.” *Both* kinds of number are “separable from sensible things.”

Speusippus: “Mathematical number alone exists, as the first of realities, separate from sensible things.” The first ten integers model the universe.

Xenocrates: Form numbers and mathematical numbers are identical.

Aristotle: Forms cannot be numbers, and ten numbers are insufficient in any case. All of these doctrines are illogical.

Now the foundations of mathematical logic present enormous difficulties, not all of which are settled in the twentieth century. Today it is easy to agree with Aristotle that his rivals could not generate numbers in a logical manner; their disputes direct our attention to the roles of the first ten integers as Platonic forms, i.e. , as Socratic “children up to ten” *(Republic* 540) upon which all the hopes of political philosophy once rested.[4] Aristotle's criticism must have seemed unanswerable to Platonists, for the doctrine of form-numbers eventually disappeared, at the cost of rendering all of Plato' s mathematical allegories incomprehensible. The role of music, now hidden in Plato, was devalued for others. Number and tone lost their privileged status and mathematical physics ceased to be important to philosophy. From Plotinus (3rd c. A.D.) onwards, Platonism shows an excessive pride of spirit together with a contempt for the physical world, diametrically opposed to Plato's reverence for it.

Although form-numbers were officially excised from Western philosophy, they still continued to play their ancient roles within the underground of esoteric religious tradition. Form numbers inspired the ancient Hebraic mystical meditation in *Sepher Yezirah.* They gave great inner confidence to the gnostic religions. Modern cabalists still pay fervent respect to the first ten integers.[5] The *musical* implication which originally justified form numbers, and their significant trigonometrical functions explored in chapter 9, are never mentioned. In Italy the multiplication table to 10 × 10 is still called the Pythagorean table, and our worldwide conversion to a decimal system will enshrine the first ten digits in still a new way for some time to come. With the adoption of the Hindu-Arabic decimal notation from 0 to 9, however, the “holy Decad” of Philolaus and the Pythagoreans suffered a radical shift of meaning, no doubt related to the vigorous resistance the West made to the introduction of this superior notation. The “tension of existence out of nonexistence,” as Eric Voegelin phrases it, had given a very special aura to the first number, 1, symbolizing the deity by its perfect and indivisible clarity.[6] To adopt a system of symbols in which 1 is preceded by zero must have seemed like a threat to God himself. The adoption of zero and a “place-value” system which has no room for 10 among the primary digits forced the cabalists to revise their meditations on numbers and now confounds modern efforts to unravel their tradition.

In summary, we need a major study of the continuing role of Platonic form-numbers in the West, and a careful search for their roots in the Hindu-Semitic East.

CRANTOR

The earliest treatise devoted to a Platonic dialogue is that by Crantor, a pupil of Xenocrates, on *Timaeus,* c. 320 B.C.[7] Fragments which survived to Plutarch's time, fortunately preserved by him, teach us how the numbers from which the World-soul was generated should be arranged in the form of a lambda (cf. figure 23). Since Plato specifies not a lambda but a chi (χ), Crantor thus helps us restore the table to its original form by taking reciprocals of the lambda for the upper half of the chi. The use of 384 as multiplier for the diatonic scale, easily ascertained in any case, can be traced to him. Crantor's objections to Xenocrates' teaching that the Platonic World-Soul was, or was like, a "self-moving number" has kept that issue alive. In figure 25, which notates all the *Timaeus* tones as powers of 3, we see what Xenocrates may have intended. Aristotle's observation that Pythagoreans generated the universe from the number 3 helps us understand Xenocrates' metaphor, despite Aristotle's contempt for his friend's language.

ARISTOXENUS

Aristotle's pupil, Aristoxenus, "The Musician" to the ancient world for the next several centuries, was notably contemptuous of both Socrates, and Plato. Aristoxenus opens the discussion in *Harmonics,* his sole surviving treatise, with an interesting condemnation of his predecessors in the field of music theory: a) "They selected for exclusive treatment a single magnitude . . ., namely , the Octave"; b) they started with the interval of the fourth and then moved "both upwards and downwards"; and c) they aimed at compression in their endeavor to exhibit "a close-packed scheme of scales"[8] (Book I, 2-8). Aristoxenus' complaints are a catalogue of Plato's virtues. Aristoxenus' observation that discords—he particularly mentions the tritone—have an *appreciable locus of variation,* as opposed to concords like the fifth 2:3 and fourth 3:4 which have "an inappreciable locus of variation," opens the door to temperament (Book II, 55). The Aristoxenian viewpoint alleviates the tyrant's suffering by ignoring commas (cf. fig. 10), and reinforces Aristotle's conception of a “relaxed proportion.” Intervals less than one-sixth of a tone (e.g., c. 200/6 = 33 cents) “are not melodic elements” to Aristoxenus; they “cannot take an independent place in the scale” (Ibid. I, 25). Western musical experience has confirmed Aristoxenus' opinions; keyboard and fretted instruments do not provide for such small intervals, and performers on unfretted strings (as in the orchestra) consciously exploit the variability of pitch, raising and lowering “leading tones” to clarify tonal functions. When Aristoxenian theory is studied against the background of Plato's mathematical allegories, then we shall understand more clearly what it attempted to do.

EUCLID

The contents of Books 7, 8, and 9 of Euclid's *Elements* are now considered to be of Pythagorean origin, or older, and authorship of Book 8 is specifically attributed to Plato's friend, Archytas.[9] So too is authorship of the *Sectio Canonis,* the earliest treatise on the tuning of the scale, also traditionally ascribed to Euclid. This material constitutes the background for Plato's mathematical allegories. “Euclid” (Archytas?) habitually represents numbers by line segments appropriate to monochord thinking; the unit (1) is thus infinitely divisible geometrically, although not numerically. Propositions 11 and 12 of Book 8 of the *Elements* “between two square numbers there is one mean proportional,” and “between two cube numbers there are two mean proportional numbers”—are the foundations for Plato's *Timaeus* Chi (X) and for all Nicomachean tables (cf. chapter 5). Book V of the *Elements* contains the Eudoxian theory of proportions, generalized to cover the continuum which line numbers” always suggested but which Pythagorean rational numbers could not exhaust. That material was developed, possibly in the Academy, during Plato's Pythagorean period and no doubt contributed to his acceptance of irrationals as *numbers* in their own right.

The last two propositions of the *Sectio Canonis,* numbered 19 and 20, describe the monochord division of the so-called “Greater Perfect System.” [10] All tetrachords are identical with those in the *Timaeus* World-Soul; octaves are similarly identical. (In the accompanying diagram I have added modern equivalents to the Greek names which obscure this identity). We have seen how Plato's aim at compression would reduce such redundancy to a single cycle, although Greek musical theorists continued to use a two-octave range for all the transpositions and modal permutations of their material. The ground-tone of the string was considered to lie outside the pattern of symmetrical development

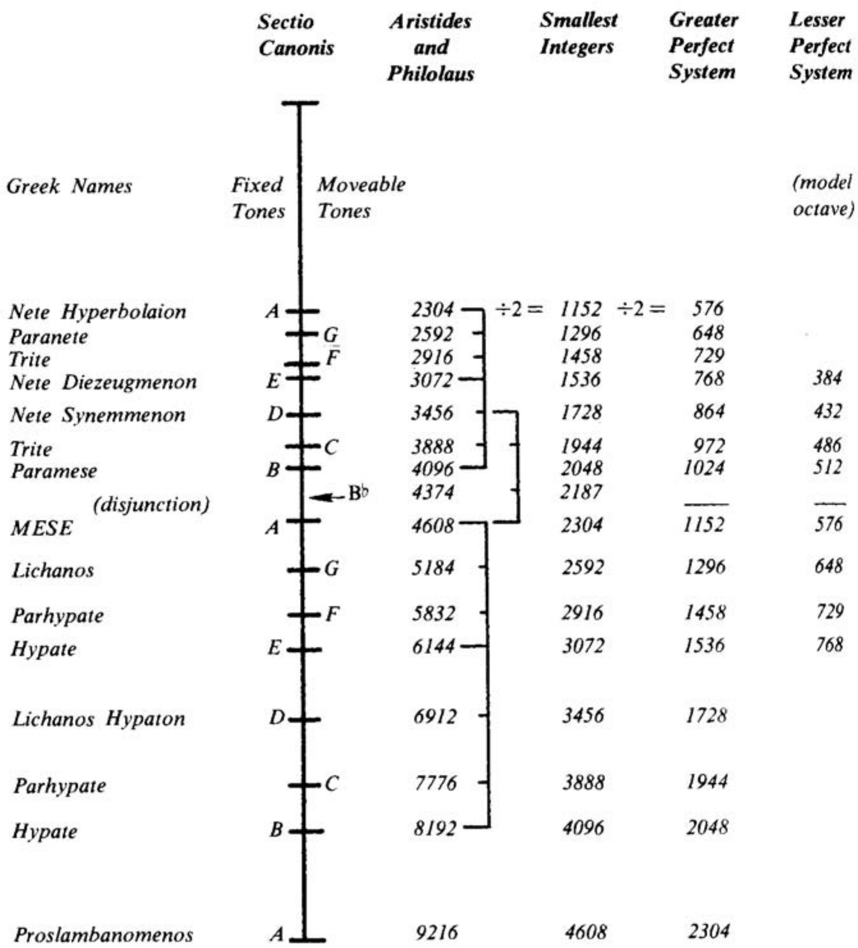


FIGURE 45

The Greater Perfect System of the *Sectio Canonis*

Numerical notation and modern alphabetal notation have been added to the Greek names, along with an optional B♭ needed for a conjunct tetrachord above A = MESE. All tetrachords have the *Timaeus* ratios: 192:216:243:256. Smallest integers for the Greater Perfect System must be doubled to provide for the optional B♭, and re-doubled for Philolaus' computation of the number of commas within the wholetone disjunction 8:9 = 4096:4608.

around the central octave, and was called Proslambanomenos, “added tone.” This is the tone which lies outside Plato's “planetary system” and symbolizes the great circle of fixed stars (cf. figs. 16-19). To avoid the use of accidentals, this tone is generally labelled A in modern notation, and so is *Mese,* “middle tone of the string. The central octave thus appears as descending from E. In my notation all three of these tones—Proslambanomenos, Mese on A, and starting tone E—become cyclic identities on D. Notably absent from the *Sectio Canonis* are the reciprocals which characterize all of Plato's own constructions. Archytas (assuming him to be the author) has projected a linear order consonant with the Pythagorean assumption of a contrast between "One" and "Many," in sharp conflict with Plato's constructions which assume a contrast between "One" and "the Great and the Small." In brief: the difference between the *Sectio Canonis* and the World-Soul, i.e., between Archytas and Plato, reflect the differences Aristotle noted between Pythagorean and Platonic views on numbers.

PLUTARCH

Perhaps the most important single clue to the sexual metaphor in Plato's mathematical allegories and a sadly misleading one—is to be found in an essay by Plutarch (c. 50-120 A. D.) on "Isis and Osiris" (373-374).[11] After reminding us that Plato identifies "father" with idea, or example, "mother" or "nurse" with material, and "offspring" or "generation" with the result, Plutarch mathematizes these notions by the 3:4:5 right triangle, leading later Platonists to assume that Socrates' marriage numbers were meant to be understood only geometrically, rather than *musically,* in the broadest sense of that term. Plutarch identifies the "upright 3" as male, the "base 4" as female, and the "hypotenuse 5" as "the child of both." He then notes that the female 4 is a square whose root is 2; this permits him then to describe the child 5 as "in some ways like to its father, and in some ways like to its mother, being made up of 3 and 2." By explicitly linking the prime numbers 2, 3, and 5 with their actual roles in Platonic genetic theory, Plutarch has handed us a key not only to Plato's mathematical allegories but probably also to the numerology in the mythologies of several older civilizations.

Plutarch's essays are an important source of ancient number lore and miscellaneous musical practices, but a major scholarly effort will be required for that material to be collated and understood in a coherent way. As a priest in the cult of Isis he displays a certain love of secrecy. As a writer he exhibits a love for exact detail, an addiction for digression, and such enthusiasm for telling stories that it is doubtful if he could keep a secret. It is seldom clear how much he understands of what he reports. In the same essay in which he casually mentions his valuable information on Plato's marriage allegory, Plutarch includes many data on Egyptian mythology. The Egyptians, he writes, consider the number 60 as "the first of measures for such persons as concern themselves with the heavenly bodies," mentioning also that Egyptian crocodiles, which live 60 years, also lay 60 eggs which hatch in 60 days.[12] Is this metaphor, intended to suggest a sexagesimal expansion? We are lucky that Plutarch preserved Crantor's clues to the World-Soul for us. It is impossible to guess how much more he may teach us when historical studies are further developed.

NICOMACHUS

Our most important guide to early Pythagorean number theory is the *Introduction to Arithmetic* by Nicomachus (fl. c. 100 A.D.), intended in part as an introduction to Plato's mathematical allegories.[13] Nicomachus quotes extensively from the *Republic, Timaeus,* and *Laws* while leading us systematically into an understanding of numerical relations and the metaphor in which they are described. It is a pity that Nicomachus' essay on Plato's marriage allegory, which he promises the reader (I.24.10), was either lost or never written, for there can be little doubt that he understood Plato's calculation and its meaning.

Nicomachus begins by defining arithmetic as "absolute quantity" and music as "relative quantity" (I.3.1). He notes that the "absolute," arithmetic, is *logically* prior to the "relative," music (I.5.1). His exposition concentrates on the first ten integers, "the ten arithmetical relations for a first *Introduction"* (I.23.4). He knows nothing at all of zero: "It is not possible to subtract 7 from 7" (I.13.13). He describes prime numbers as "a fountain and a root" for all other numbers (I.11.3). He associates unity with equality and sameness, and then produces all species of inequality (reminiscent of Plato's "dyad of the Great and the Small") out of equality, first and alone, as from a mother and root" (I.23.6). Nicomachus is so pedantically rigorous in pursuing absolute symmetry, assigning appropriate technical terms to every relation and to its reciprocal, always studying numbers by "the use of direct and of reverse order" (I.23.13), that today he is accused of "a perverse anxiety to be symmetrical."[14] That accusation exposes his sympathy for, and our oversight of, Pythagorean habits.

Nicomachus notes that Platonic psychogony requires us to know how to write a series of numbers in "continued geometrical proportion" *(Timaeus* 31c), such as 4:3, 3:2, 2:1, etc. (II.2.3), these series to contain 2, 3, 4, 5 "or an infinite number" of terms. (The *Timaeus* arithmetic actually requires us to determine ten such terms, shown in the Nicomachean table of figure 22.) Such series are limited by *multiples* of the generators, and Nicomachus states the rule for finding the *smallest* terms needed:

> *Every multiple will stand at the head of as many superparticular ratios corresponding in name with itself as it itself chances to be removed from unity, and no more nor less under any circumstances (II.3.1).*

Such multiples are "father" to the sequence of numbers which can follow. (The rule also applies to ratios like 5:3, for instance, which are not *superparticular,* but Nicomachus, like Plato, is concerned primarily with pairs of numbers whose difference is unity, 1.) Unity, having "the character of a point, will be the beginningof intervals and of numbers, but not itself an interval or a number" (I.6.3).

Nicomachus expounds for pages on Sameness and Difference, linking the first with odd numbers and the second with even (II.17-20). He defines harmony as "the reconciliation of the contrary-minded" (II.19.1). He regards 10 as a larger "unit" and 100 as a still larger "unit," reflecting the abacus habits of an earlier generation (I.19.17).[15] Square numbers display more of "sameness" (linked to odd numbers, and through them to the primal unity, 1) than do the products of other numbers, and cube numbers display this prized quality the most of all (1I.20.1-3).

Nicomachus tells us explicitly how to construct the multiplication table for 10 × 10, and then describes all the different ways we must look at this table to comprehend its layers of potential meaning. The pedantry of his exposition is forced on him in part by the Greek alphabetical notation then current, and whose obscuring character he deplores (11.6.2). In figure 46 the notation which hindered Nicomachus is shown together with its modern Hindu-Arabic equivalent, which allows many arithmetical patterns to be seen at a glance. Nicomachus finds all of Plato's ratios in this table and no superfluous ones—in patterns of the Greek letters gamma Γ and a slightly rotated chi χ, important to the *Timaeus* imagery.

To share Nicomachus' way of seeing ratios in the multiplication table, note that the first row contains "the natural series from unity," and successive rows contain its multiples (I.19.10). But the natural series from 1 through 10 occupies not only the first row but also the first column , "in the form of the letter gamma" Γ (I.19.11). Such numbers form a sequence of superparticular ratios (n + 1:n, such as 2:1, 3:2, etc.) and their sub-superparticular converses (n:n + 1, such as 1:2, 2:3, etc.). In fact, the whole table is to be read as a nest of

α	β	γ	δ	ε	ς	ζ	η	θ	ι
β	δ	ς	η	ι	ιβ	ιδ	ις	ιη	κ
γ	ς	φ	ιβ	ιε	ιη	κα	κδ	κξ	λ
δ	η	ιβ	ις	κ	κδ	κη	λβ	λς	μ
ε	ι	ιε	κ	κε	λ	λε	μ	με	ν
ς	ιβ	ιη	κδ	λ	λς	μβ	μη	νδ	ξ
ζ	ιδ	κα	κη	λε	μβ	μφ	νς	ξγ	ο
η	ις	κδ	λβ	μ	μη	νς	ξδ	οβ	π
θ	ιη	κζ	λς	με	νδ	ξγ	οβ	πα	ϙ
ι	κ	λ	μ	ν	ξ	ο	π	ϙ	ρ

a) Greek notation

1	2	3	4	5	6	7	8	9	10
2	4	6	8	10	12	14	16	18	20
3	6	9	12	15	18	21	24	27	30
4	8	12	16	20	24	28	32	36	40
5	10	15	20	25	30	35	40	45	50
6	12	18	24	30	36	42	48	54	60
7	14	21	28	35	42	49	56	63	70
8	16	24	32	40	48	56	64	72	80
9	18	27	36	45	54	63	72	81	90
10	20	30	40	50	60	70	80	90	100

b) Hindu-Arabic notation

FIGURE 46

The Multiplication Table for 10 × 10

The twenty-seven letters of the Greek alphabetical notation are contrasted here with the ten symbols of Hindu-Arabic notation. [16]

gammas , square numbers functioning as the “greater unities” (I.17.3). From these numbers lines radiate in sequences of superparticular and sub-superparticular ratios, in multiples of their smallest integer forms. Within this table Nicomachus finds many species of multiples and submultiples, assigning labels to symmetric pairs in tedious detail which maintains perfect inverse symmetry among these potential tone-generators. He asks us to observe how the letter chi χ emerges among these ratios when any gamma Γ is extended backwards to the outer rows. 

Nicomachus explicitly declares these numbers to be relevant multiples and submultiples of lengths on both strings and pipes (I.27.1). As divisors of lengths, Nicomachean integers define harmonic series based on any unit reference length; as multiples of length, Nicomachean integers define the reciprocals of such harmonic progressions. It is obvious that he is reading into the same multiplication table the interlocking sequences of arithmetical and harmonic progressions, correlating with rising-falling sequences of pitches. It is with a touch of genius that Albert von Thimus reconstructed the Pythagorean multiplication table in a way which renders the old tonal implications immediately visible to us.[17] Thimus writes the table in modern fractions and their reciprocals, adding the tone-values men like Nicomachus took for granted. In Thimus' form of the table, shown in figure 47, the fractions are the ratios between two integers whose products give the integers in Nicomachus' table. Every row is an harmonic series (from 1 through 9); every column is a reciprocal harmonic series; the square numbers which constitute Nicomachus' “greater unities” have the form 2/2, 3/3, 4/4, etc., all equal to 1.

Whatever its actual historical status, the Thimus table is a provocative presentation of fundamental Pythagorean tonal-arithmetical thinking. Early in the Christian era the Hindus not only possessed the Thimus fractional notation but also habitually wrote integers in the form of nil (presumably as an ever-present reminder that they are multiples of unity). Furthermore, the Hindus, like Thimus, extended the number series backward to n/o and o/n, even to the infamous 0/0 which Greek thinking rejected.[18] For Thimus, 0/0 is the graphical point of origin for a geometric interpretation of string lengths. This symbol gives a very beautiful meaning to the notion of an “unknown god,” about whom absolutely nothing can be predicated, standing behind the Greek-Hebraic god of unity = One, who *functions* as the creative demiurge of the actual world. That such an idea had been trying to enter Greek rationality for centuries is proven by Aristotle's need to deny absolutely that the *void* can be part of ratio theory: “The Pythagoreans . . . held that void exists and that it enters the heaven itself...(and) distinguishes the

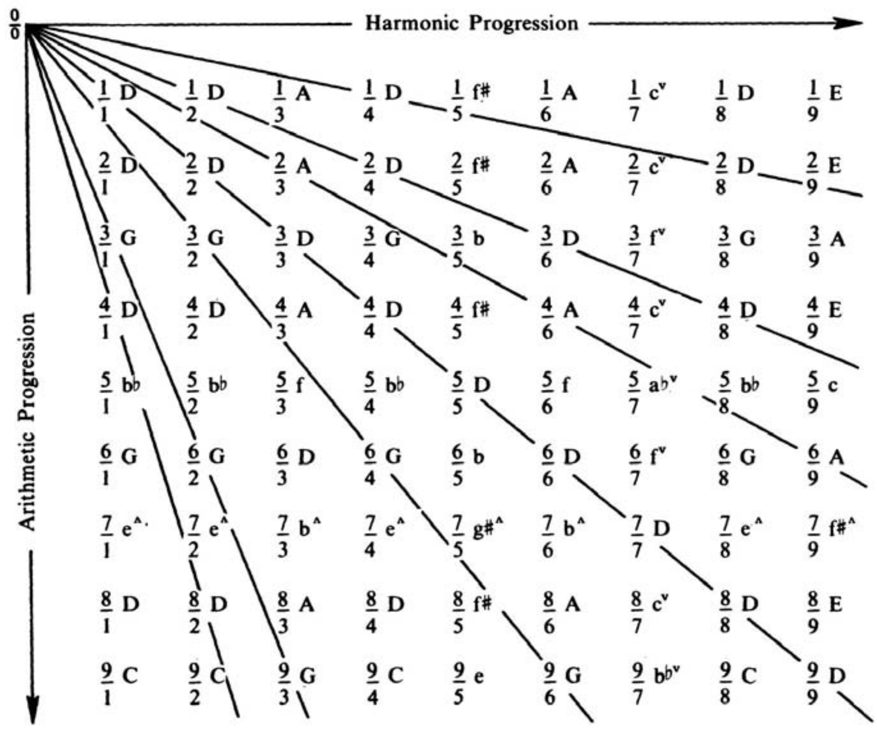


FIGURE 47

The Pythagorean Table of von Thimus

Products of numerator-denominator pairs are the integer values of figure 46. “Identity rays” meet planimetrically at a point which can be considered 0/0. The 19 different tone-values in the table are all guardians in Plato's Magnesia. (This is an, adaptation of figures 22 and 24 in *Tone: A Study in Musical Acoustics,* by Siegmund Levarie and Ernst Levy.)

natures of things, as if it were like what separates and distinguishes the terms of a series” *(Physics* 213b), but, for Aristotle, “there is no ratio of 0 to a number” *(Physics* 215b). Did such irrefutable logic banish from fourth century Greece a mode of thinking which von Thimus recovered? This question must be left to historians of science to answer when they are ready.[19]

Every element in the Nicomachus-Thimus table is a guardian in Plato's Magnesia. All doubles are octave equivalences and actually coincide in Platonic tone-circles. I have omitted the tenth row and tenth column, double or half, respectively, of the fifth row and fifth column, so that my table stops at 9 × 9 = 81. These 81 numbers, tonally equivalent to Nicomachus' 100 numbers, have exactly 19 different tone-values belonging to the following Platonic "classes" (cf. figs. 33-37):

First class (1:2, 2:3, 3:4): D, plus the twins A-G and C-E.

Second class (4:5, 5:6): twins c - e, f - b, and f♯b♭.

Third class (6:7): twins c^{v} - e^, b♭^{v} - f♯^, a♭^{v} - g♯^, and f^{v} - b^.

At the beginning of the Christian era the Hindus were apparently still calculating with just nine symbols.[20] The square of 9 (= 81) plays a significant role in ancient Chinese numerology, and an even more important one in the earlier Babylonian sexagesimal arithmetic. The largest integer which generates standard Babylonian tables of reciprocals is 81, written as 1:21 and meaning both 60 + 21 and 60:81, since 60 = 1 is always the point of reference. Its sexagesimal reciprocal is written 44:26:40, meaning $44 \times 60^2 + (26 \times 60) + 40 = 160{,}000$; if conceived as the reciprocal of $^{81}/_{60}$, it must be thought of as 160,000/216,000, such multiplications and divisions by the powers of 60 having no effect on the written form of the number, whose precise meaning can always be determined from the context. Converting these important Babylonian limits to their simplest decimal equivalents teaches us a possible new implication for Socrates' "sovereign" number 12,960,000:

$$\frac{81}{60} \times \frac{160{,}000}{216{,}000} = \frac{12{,}960{,}000}{12{,}960{,}000} = 1$$

Is it possible that Pythagoras actually brought the material of the marriage allegory home from his travels in Babylon?[21] The "Babylonian" aspects of the marriage allegory have been debated a very long time without consideration for the musical aspects of the argument. Again, historians of science must be granted the last word, but not until they have weighed all of the musical evidence of which they seem so far to have been unaware.

Nicomachus was copied devotedly in Europe for the next thousand years while the newer mathematics of Theaetetus, Eudoxus, and their successors was forgotten. This extreme conservatism ought to have ensured that many people continued to understand Plato's kind of arithmetic, but such was not the case because, except for parts of *Timaeus,* his dialogues were lost to Western history for the next thousand years or more. We must remain alert to the possibility, however, that ancient authors whom we now misunderstand did continue to keep alive valuable insights into Plato's musical allegories, for their Nicomachean "key" was never lost.

PTOLEMY

Our oldest extant tone-circle is the tonal zodiac of Claudius Ptolemy (fl. 125-148 A.D.).[22] He bends the two octaves of the Pythagorean scale in the *Sectio Canonis* "round into a circle" in Plato's manner so that beginning and end coincide (cf. figure 48). Since Ptolemy postulates 12 *equal* divisions while defining each one as a wholetone 8:9, he has bequeathed us not only the oldest example of "octave equivalence" but the first example of equal-temperament.[23] His zodiac is a two-octave tempered wholetone scale, with equivalent octave-semicircles and with Mese (middle) associated with Libra (Balance).

Ptolemy's writings are the source of much of our knowledge of Greek musical theory. He recorded the ratios of some twenty tunings, including several of his own, and considered the syntonic comma 80:81, by which some of them differed, as acoustically neglible.[24] Ptolemy's transmission of the Archytas tunings for the diatonic, chromatic, and enharmonic octave genera is of the utmost importance in helping us understand Plato's possible sources. They are shown below in figure 49 both "close-packed," as Aristoxenus described his predecessors' work, and as separate octave-sequences. Notice that only one string in each Archytas tetrachord is "movable," as opposed to two in later Aristoxenian theory, thus simplifying the problem of retuning lyre and kithara when changing genus. Archytas has simply added pure thirds 4: 5 (at D-b♭ and G-e♭) and septimal seconds 8:7 (at C-b♭^{v} and F-e♭^{v}) to an underlying Phrygian octave in Pythagorean tuning.[25] (In figure 49 I have tried to suggest how Archytas could have proceeded, using the *Sectio Canonis* now attributed to him as a model.) The novel element in Archytas' tuning is apparently the introduction of the prime number 7 as a "tone-generator," providing thereby the model for Plato's Magnesia. The tones I have labelled B and F are missing in Plato's adaptation, and much of Plato's inverse symmetry is missing from the Archytas system. An arithmetical analysis suggests why Plato could not simply use his

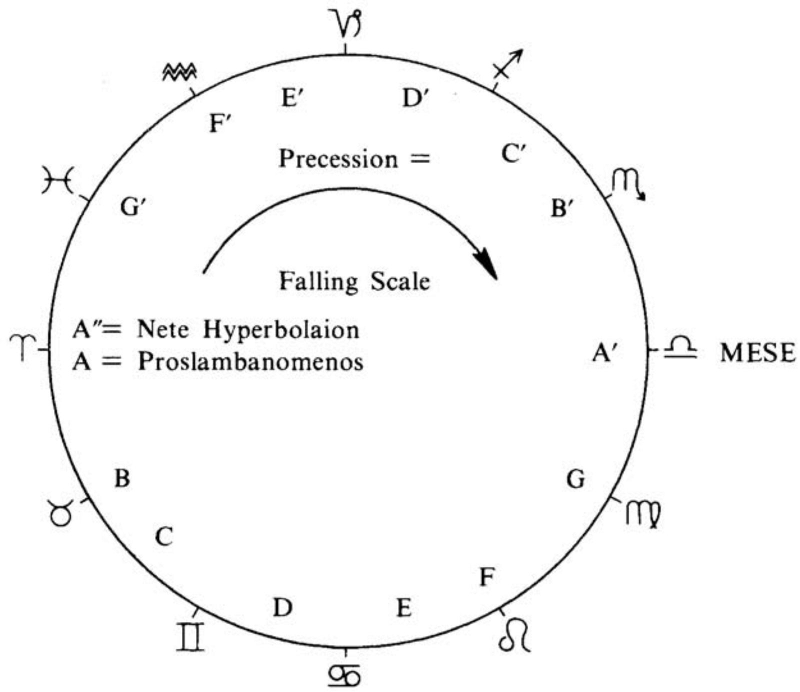


FIGURE 48

Ptolemy's Tonal Zodiac

The twelve signs of the zodiac are correlated with the fifteen tones of the two-octave Pythagorean scale on which was established the so-called Greater Perfect System of ancient Greece. The ground-tone of the monochord string, *Proslambanomenos,* and the highest tone, *Nete hyperbolaion,* two octaves above, coincide. I have substituted the usual modern letters for Ptolemy's Greek names for the tones. Since each wholetone is 1/12th the circumference of the circle, his zodiacal signs actually correlate with a wholetone scale in equal-temperament (after Ingemar During).

Tetrachords A G F E D C B

friend's work as it stands for a model of a "practicable" city: the numerical "index" for the Archytas system is 4,320 (providing smallest integer names for all tones, as ratios of frequency); to provide inverses for the septimal tones $b\flat^{v}$ and $e\flat^{v}$ would require the index to be multiplied by 7 (e.g., $7 \times 4{,}320 = 30{,}240$), resulting in an impracticably large "Socratic population." The smallest integers for Archytas' string length

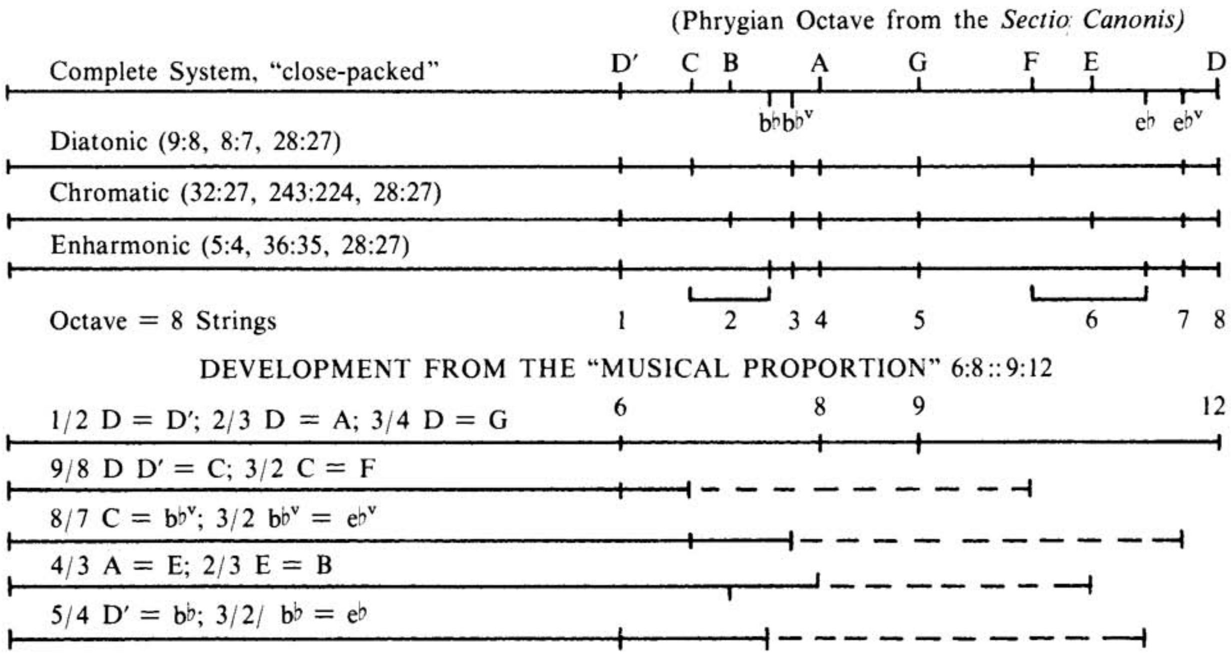


FIGURE 49

MONOCHORD ANALYSIS OF ARCHYTAS' TUNINGS

Capital letters F C G D A E B are tones in Pythagorean tuning, linked by perfect fourths and fifths; b♭and e♭are Just or "pure" thirds; b♭ᵛ and e♭ᵛ are "septimal" tones characteristic of Archytas.

ratios require an index of 6,048; the smaller numbers given here actually function as frequency ratios.

FREQUENCY RATIOS

Composite

D'	$2^5 3^5 5$	= 4320
C	$2^8 3 \cdot 5$	= 3840
G	$3^6 5$	= 3645
b♭	$2^7 3^3$	= 3456
b♭^{v}	$2^5 3 \cdot 5 \cdot 7$	= 3360
A	$2^3 3^4 5$	= 3240
E	$2^6 3^2 5$	= 2880
F	$2^9 5$	= 2560
D	$2 \cdot 3^5 5$	= 2430
e♭	$2^8 3^2$	= 2304
e♭^{v}	$2^6 5 \cdot 7$	= 2240
D	$2^4 3^3 5$	= 2160

Ptolemy is an important link not only to earlier developments in Greece but to possibly still earlier ones in Babylon. It is not yet clear how much either his astronomy or his geometry owes to Babylonian methods, but his arithmetic procedure is suggestively archaic. Ptolemy linked the monochord on which he recorded his tunings to the diameters of the great circles in which he projected his astronomy, taking the unit radius as 60, the diameter as its double at 120, and then subdividing these units into 60 minutes of 60 seconds each. This procedure links his unit radius to Plato's sexagesimal expansion from 60 through 60^2 to 60^3 = 216,000, and links its double at 432,000 to the Hindu "Kali Yuga" time period, a number which recurs in various ways in old-world mythology. Ptolemy's work on music, mathematics, and astronomy is a pivotal center in the meeting of East and West, and in the transformation of the ancient world into the modern one. We have scarcely begun to understand him. His own links to Plato's mathematics are far from clear: Socrates' sovereign number is 360 × 36,000 = 12,960,000, and there has been considerable speculation that Ptolemy took the factor of 36,000 as the period of the precessional cycle (a figure substantially too large and less accurate than an earlier estimate by Hipparchus). Ptolemy also computed trigonometric values based on ½ degree, "the length of a side of a polygon of 720 sides inscribed in a circle of radius 60 units." [26] We have noticed in the marriage allegory (cf. fig. 6), and again in *Laws* (cf. fig. 34), how Plato uses an octave double 360:720 containing 360 *arithmetical* subdivisions in its cycle; Ptolemy's circles contain two such octaves (each semicircle being an octave). These figures are derived from Just tuning, not from the Pythagorean tuning which Ptolemy projected into his tonal zodiac; but he himself considered the acoustical differences to be negligible. It might be a mistake to conclude that either music or Plato had a direct influence on Ptolemy's geometry, but it is also evident that in Ptolemy's mind all of these matters were correlated.

ARISTIDES QUINTILIANUS

The only clues we possess to the Greek modes in the time of Damon, the Athenian musician whom Socrates was fond of quoting, are a set of tunings preserved by Aristides Quintilianus (probably 3rd or 4th c. A.D.).[27] They are presumably derived from a lost work of Aristoxenus; their authenticity cannot be confirmed. They are interesting to musicians because of their many anomalies; not one of them fits the tetrachordal theory of Aristoxenus—a possible argument in favor of their antiquity. I present them here as they are sounded by the guardians of Magnesia in order to suggest exactly how, in Plato's time, Dorian and Phrygian "employ the same sounds," as Aristotle affirmed; how all other modes are derivative from these two; and why Socrates is therefore justified in banning all modes but the Dorian and Phrygian from his very elegant model-building. In order to make the correlation in figure 50, I have interpreted Aristides' tones, ditones, and quartertones according to Archytas' ratios for the enharmonic genus (i.e., tones 8:9, ditones 4:5, and successive quartertones 35:36 followed by 27:28), for Aristides himself does not specify exact values. In figure 51 these modes are projected into a tone circle derived from that of figures 36-39.

PROCLUS

Modem interpretations of Plato's *Timaeus* are heavily indebted to an extensive commentary by Proclus (410?—485 A.D.), one of the last to head the Academy before it was closed by order of Justinian in 529, some of its faculty fleeing then to the court of the king of Persia.[28] The Academy had suffered several reorganizations during its 916 years of existence, and although it remained devoted to Plato, by the time of Proclus the construction of the World-Soul had become problematical.

DORIAN: 1, 1/4, 1/4, 2, 1, 1/4, 1/4, 2

G^1	A^1	$b\flat^{v1}$	$b\flat^{2}$	D^1	E^1	f^{v1}	f^2	A^1
3360	3780	3920	4032	5040 / 2520	2835	2940	3024	3780
	9:8	28:27	36:35	5:4	9:8	28:27	36:35	5:4

PHRYGIAN: 1, 1/4, 1/4, 2, 1, 1/4, 1/4, 1

G^1	A^1	$b\flat^{v1}$	$b\flat^{2}$	D^1	E^1	f^{v1}	f^2	G^1
3360	3780	3920	4032	5040 / 2520	2835	2940	3024	3360
	9:8	28:27	36:35	5:4	9:8	28:27	36:35	10:9

LYDIAN: 1/4, 2, 1, 1/4, 1/4, 2, 1/4

$b\flat^{v1}$	$b\flat^{2}$	D^1	E^1	f^{v1}	f^2	A^1	$b\flat^{v1}$
3920	4032	5040 / 2520	2835	2940	3024	3780	3920
	36:35	5:4	9:8	28:27	36:35	5:4	28:27

MIXOLYDIAN: 1/4, 1/4, 1, 1, 1/4, 1/4, 3

E^1	f^{v1}	f^2	G^1	A^1	$b\flat^{v1}$	$b\flat^{2}$	E^1
2835	2940	3024	3360	3780	3920	4032	[5670]
	28:27	36:35	10:9	9:8	28:27	36:35	45:32

IONIAN: 1/4, 1/4, 2, 1 1/2, 1

A^1	$b\flat^{v1}$	$b\flat^{2}$	D^1	f^2	G^1
3780	3920	4032	5040 / 2520	3024	3360
	28:27	36:35	5:4	6:5	10:9

SYNTONOLYDIAN: 1/4, 1/4, 2, 1 1/2

A^1	$b\flat^{v1}$	$b\flat^{2}$	D^1	f^2
3780	3920	4032	5040 / 2520	3024
	28:27	36:35	5:4	6:5

FIGURE 50

The Ancient Greek Modes of Aristides Quintilianus

The ratios are those in Archytas' enharmonic genus, the interval sequences are those recorded by Aristides Quintilianus, and the integer sequences are taken directly from the guardians in *Laws.*

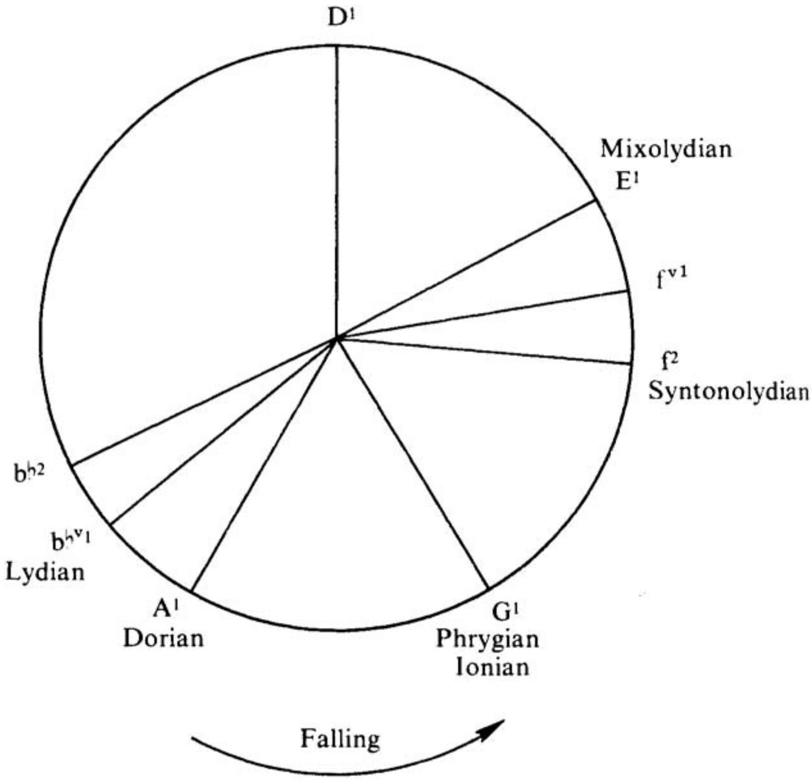


FIGURE 51

Dorian as the Comprehensive Mode

Aristides' intervals, Archytas' ratios, and Plato's integers are correlated here to show how the ancient Dorian mode comprehended all of the others. Tones are correlated with the modes in which they function schematically as the *highest pitch,* without regard to which tone may have been considered a tonal center, or Mese. (This figure must he studied together with figures 36 and 46.)

Proclus does not disguise his uncertainty over how the *Timaeus'* "portions" are to be "taken," and how many such portions there were in Plato's mind. His very extensive commentary on earlier efforts to solve these problems is guided by one clear insight: a document known as *Timaeus Locris,* now thought to be an invention of the first century A.D., contained a spurious construction limited to 36 "portions," harmonious with the cosmological significance of 36 (i.e., the Egyptian sky had been divided into 36 "decans," and the year into 36 ten-day periods), but not consonant with Plato's methods.[29] In correcting the errors of

Timaeus Locris, Proclus makes a worse error himself, but helps us discover what I believe to be the more sensible solution presented in my chapter 5.

The *Timaeus Locris* version of the World-Soul is shown in figure 52 together with Proclus' correction, and with A. E. Taylor's further correction of Proclus; each version is worse than the preceding one. All of them ignore Plato's hint that the limiting index is "roughly some twenty thousand," and all eliminate certain numbers within the sets they do use in spite of the fact that such numbers form regular Dorian tetrachords of $9/8, 9/8, 256/243$, which Plato specifies. (Please note that I have numbered the elements in figure 52 to correlate with those in figure 21; only 38 possible elements exist in this set, items 12, 25, and 40 appearing only when its numerosity is doubled.) These constructions damage the World-Soul by leaving the original portion of 27 units, the very last element in the set, standing outside the tetrachord pattern Plato specifies. The common limiting index of 10,368 actually belongs to the one-octave scale in the Spindle of Necessity (a. fig. 16), rather than to the *Timaeus* construction, if I am correct in following Severus rather than Proclus. These details are trivial in themselves; they become critical, however, in revealing the misunderstanding of Plato's planetary model. Plato initially gives us a formula which perplexes Platonists because it generates what seems like too much musical material without furnishing a rule for excluding what is superabundant, hence the *ad hoc* decisions of *Timaeus Locris,* Proclus, and Taylor. Since all possible integers $2^p 3^q \leq 20{,}736$ can be linked into Dorian tetrachords, Plato needs no exclusion principle, and his apparent vagueness is a tribute to his far-seeing vision of the arithmetical development. He himself excludes octave replications later by bending his soul-stuff, as it has been called, "round into circles," that is, by viewing it cyclically. In the second half of the allegory, he then excludes all but seven tones from any particular planetary set by subdividing his "circle of the same into exactly seven sub-circles (cf. figs. 24-28). Because Platonists know that the material must eventually fit a one-octave scale, they are tempted, like Taylor, to make exclusions in the first half of the allegory rather than let Plato make them in his own way in the second half. This is not as trivial as it may seem, for the original construction, derived from Philolaus (according to Aristoxenus), shows how "matter" degenerates into confusion (cf. the overlapping tetrachords of figs. 21 and 52) the further we progress from the celestial model in the *first* octave. As figure 52 shows, *Timaeus Locris* omits 2 of 38 possible tone-numbers, losing only octave-duplicates in the process; Proclus, however, omits 4 numbers and loses 2 of the possible tones, while Taylor omits a different 4, hut with the same effect.

	Falling	Terms	Means	Timaeus L.	Proclus	Taylor
1	D	1	6	384	384	384
2	C			432	432	432
3	Bb			486	486	486
4	A		8	512	512	512
5	G		9	576	576	576
6	F			648	648	648
7	Eb			729	729	729
8	D	2	12	768	768	768
9	C			864	864	864
10	Bb			972	972	972
11	A		16	1024	1024	1024
12						
13	G	3	18	1152	1152	1152
14	F			1296	1296	1296
15	Eb			1458	1458	1458
16	D	4	24	1536	1536	1536
17	C		27	1728	1728	1728
18	Bb			1944	1944	1944
19	A		32	2048	2048	2048
20	Ab			2187	—	—
21	G		36	2304	2304	2304
22	F			2592	2592	2592
23	Eb			2916	2916	2916
24	D	8	48	3072	3072	3072
25						
26	C	9	54	3456	3456	3456
27	Bb			3888	3888	3888
28	A			—	—	4096
29	Ab			4374	4374	—
30	G			4608	4608	4608
31	F		81	5184	5184	5184
32	Eb			5832	5832	5832
33	D			6144	6144	6144
34	Db			6561	—	—
35	C		108	6912	6912	6912
36	Bb			7776	7776	7776
37	A			—	—	8192
38	Ab			8748	8748	—
39	G			9216	9216	9216
40						
41	F	27	162	10368	10368	10368

FIGURE 52

Historical Solutions of the *Timaeus* Scale

I have restored all the missing tones in Severus' set (items 11, 19, 25, 28, 33, 37, and 39 in figure 21, a total of 7 numbers, but no new tones since they are merely octave-duplicates), and have tried to let Plato abstract his 7-element planetary system from them in his own way.

If Proclus failed to understand all the arithmetic details of the World-Soul, he succeeded in the more important areas of historical thoroughness, of fairness to those with whom he disagreed, and of candor concerning his own doubts. Modern scholars like James Adam, A. E. Taylor, and Robert Brumbaugh continue that noble tradition. It is exactly at the points where such men confess their own uncertainty that a musician is alerted to examine the evidence in a new way. I am deeply in debt to their candor, and part of my purpose here is to try to correct certain misunderstandings of ancient Greek musical theory which contributed to their difficulties.

On the vexing question of Pythagorean love of secrecy, Proclus makes an interesting comment:

> *Plato, for the sake of concealment, employed mathematical names as veils of the truth of things, in the same manner as the theologists employed fables, and the Pythagoreans symbols.*
>
> *(Commentary on Timaeus,* II, 117)

Today few Platonists would agree that Plato intended to conceal anything. Our musical analysis suggests, rather, that he planted clues in abundant measure to encourage those who were capable of it to study his arithmetic in a variety of ways. To the extent that his mathematical allegories are riddles, part of the reason is that changing cultural conditions leave scholars generally ignorant of Greek musical theory; and another part of the reason is that he seems to have intended his allegories as "games" in mathematical analysis, a branch of the discipline which Aristotle considered him to have invented.[30]

BOETHIUS

Two lessons in acoustical arithmetic by Philolaus have been preserved by Boethius (480-524 A.D.).[31] Plato was accused by Aristoxenus of deriving most of his Pythagorean materials from Philolaus, teacher of his own friend Archytas. The lessons of Boethius are important clues to habits of thought in the time of Damon, Socrates, and Plato. The habits are *not* Plato's. This contrast is important.

The first lesson concerns the interval 243:256 which completes the *Timaeus* tetrachords, and which Plato calls a leimma ("left over") and Philolaus calls a diesis:

(8:9) (8:9)
192 : 216 : 243 : 256

Notice that 256—243 = 13. Philolaus uses this undersized "semitone" (worth about 89 cents instead of 100 cents) to locate, almost precisely, an equal-tempered semitone. He notes that the neighboring wholetone 243—216 = 27, that 216 + 13 = 229, that 243 — 13 = 230, and thus isolates the unit between 229 and 230 which contains $\sqrt{9/8}$, precisely half the Pythagorean wholetone 9:8, a somewhat larger value than the equal-tempered semitone, which Pythagorean arithmetic had no means of identifying (i.e., it is actually $\sqrt[12]{2}$) . Because Philolaus seems to he concerned with the mystical qualities of the numbers, both Boethius and modern scholars disparage what ought to be viewed as a simple and elegant example of Diophantine approximation. Philolaus is aware that his successive arithmetical divisions of 13 units are not equal ratios, but he has discovered an ideal context and the smallest integers possible for showing the discrepancy between the Pythagorean value 243:256 and the ideal value at which it aimed. (Using somewhat more careless language, we can say that the equal-tempered semitone is approximately 17:18 and its Pythagorean counterpart is approximately 19:20.)

The second lesson which Boethius attributes to Philolaus is an arithmetical—not a proportional subdivision of a wholetone into 9 commas. The flagrant inaccuracy of his procedure reaps the scorn of Boethius and of modern scholars alike, and there is reason to believe that Boethius has correctly reported what *appears* to be a misunderstanding of acoustical arithmetic on the part of Philolaus, one of the most prestigious Pythagoreans in history. The implications of the Philolaus construction are displayed in figure 53 not to demonstrate his stupidity, but to prove his intelligence. I believe Pythagoreans of the fifth century B.C. could casually accept integer Diophantine approximations to values they could not calculate, and they were experts in discovering the smallest, integers which served this purpose. The smallest wholetone 8:9 within which the standard leimma 243:256 (Philolaus' *diesis)* can be inserted is 2048:2304, as shown in figure 45. A simple doubling makes his further arithmetical subdivision available. The truth is that a wholetone 8:9 is slightly less than 9 Pythagorean commas and slightly greater than 9 syntonic commas. Since 9 syntonic commas 80:81 can be displayed only between 80^9 and 81^9 (cf. the rule of Nicomachus,

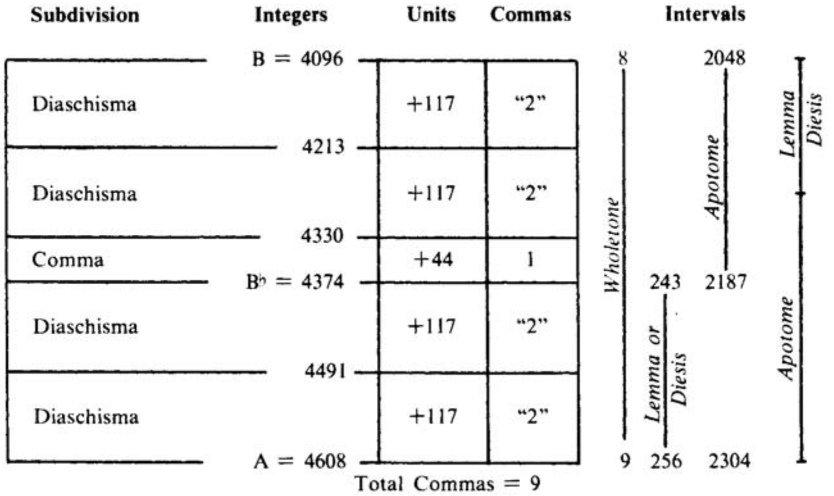


FIGURE 53

Philolaus' Division of the Wholetone Into 9 Commas

The division is carried out *arithmetically:* two intervals sharing the same name are actually not the same *proportionally,* nor are the diaschismas of 117 units actually twice the comma of 44 units. This is a "Diophantine" approximation, in smallest integers, of a deeper truth that there are indeed approximately 9 Syntonic or Pythagorean commas in a wholetone of 8:9. Philolaus further subdivides the comma of 44 units into 2 schismas of 22 units each, the smallest subdivision recorded among ancient Greek theorists:

Wholetone 8:9 ≈ 204 cents.

Pythagorean comma ≈ 24 cents, × 9 ≈ 216 cents.

Syntonic comma ≈ 22 cents, × 9 ≈ 198 cents.

supra, and the examples of the tyrant's allegory and the Sectio Canonis in chapter 3), proof that a wholetone 8:9 actually contains 9 of them depends upon demonstrating that $81^9 \le \left(\frac{9}{8} \times 80^9\right)$.

$$81^9 = 150{,}094{,}635{,}296{,}999{,}121 \text{ (= ninth comma)}$$

$$\frac{9}{8} \times 80^9 = 150{,}994{,}944{,}000{,}000{,}000 \text{ (= wholetone)}$$

Did anyone ever carry out this barbaric calculation? Probably, because the correct notion of 9 commas cannot be proved by Philolaus' crude arithmetic.[32] Instead of heaping scorn on Philolaus, as we have since the time of Boethius, we should probably award him a prize for a Diophantine simplification.

Philolaus further subdivided his comma of 44 arithmetical units into two schismas of 22 units each, the finest subdivision of the tonal continuum recorded by any ancient theorist. This schisma has an actual logarithmic value of 8 to 9 cents, and thus lies at the very threshold of audibility. Boethius, **I** suggest, has preserved for us the specific demonstration which prompted Plato's jest about men who "put ears before the intelligence" by trying to determine "the smallest interval by which the rest must be measured" *(Republic* 531). Philolaus carried that effort to its limit, and without any concern for the prime factors of the numbers he needed. Socrates properly insists that absolute truth must be pursued by the abstract methods of number theory, forcing the mind to move into a region where the evidence of the senses is no longer trustworthy, and clearly recognizing that ideal values must be mediated by philosophical common sense in the actual world. Philolaus helps us understand the tension between Platonic ideals and Platonic wisdom in expecting something less than perfection in the world we live in.

SUMMARY

From Philolaus in the fifth century B.C. through Plato and Aristoxenus in the fourth, and down to Ptolemy in the second century A.D. ano Aristides in the third or fourth, Greek acoustical theorists moved confidently between two modes of expression: the absolutely precise, and thy conveniently approximate. They were masters of their own arithmetical tools, and not the fools we have sometimes taken them to be. It is easier to understand Plato when we view him within the context of an 800-year tradition of Greek musical theory, but it is also easier to understand other Greek theorists when we understand Plato's mathematical allegories. There is urgent need for a review of all of these ancient materials not simply for their intrinsic interest to musicians and to historians of science, but for their wider relevance to the philosophical foundations of Wester_n culture.

The twentieth century has given birth to a new discipline—the study of the history of science—whose remarkable achievements change our views of the ancient world faster than we can adjust our thoughts. This is an exciting moment to live, but a poor one in which to be dogmatic about the past. The reader will understand, given the wrong notions we have inherited and the confusion in which we live, why my own dialogue with friends has been the most important factor in my Platonic studies. Siegmund Levarie always sensed Plato's intrinsic musicality; Hugo Kauder saw Pythagorean tuning as a Greek temperament; Ernst

Levy understood Pythagorean reciprocity, and saw the point of Socrates' marriage formula without having to look at any of the arithmetic details; Antonio de Nicolás grasped the reason why music was once essential to philosophy. My friends have been worth more to me than all other sources combined.[33]

Socrates would have understood. He trusted his private daemon to keep him from a wrong action; but for learning, so he claimed, he required a dialogue with friends.

Appendix II
Conversion Tables

The intervals of central concern are displayed here in three ways: (1) as the ratios of integers, (2) as logarithmic cents, and (3) as degrees in a one-octave tone-mandala.

To convert ratios to cents, subtract the log of the smaller number from the log of the larger, then multiply by 1200/log 2 (≈3986).

To convert cents to degrees, multiply by 360/1200 = .3.

All conversions are *approximations* except that of the octave, ratio 1:2 = 1200 cents = 360 degrees.

SUPERPARTICULAR RATIOS

Ratios	Intervals	Cents	Degrees
1:2	octave	1200	360.
2:3	perfect 5th	702	210.6
3:4	perfect 4th	498	149.4
4:5	major 3rd	386	115.8
5:6	minor 3rd	316	94.8
6:7	septimal 3rd	267	80.1
7:8	septimal 2nd	231	69.3
8:9	major tone	204	61.2
9:10	minor tone	182	54.6
15:16	just diatonic semitone	112	33.6
17:18	approximation to equal-tempered semitone	99	29.7
19:20	approximation to Pythagorean diatonic semitone of 243:256	89	26.7
24:25	just chromatic semitone	70	21.
35:36	approximation to equal-tempered quarter-tone	49	14.7
73:74	approximation to Pythagorean comma of 524288:531441	24	7.2
80:81	syntonic comma	22	6.6

OTHER RATIOS

64:81	Pythagorean "ditone" third	408	122.4
243:256	Pythagorean *leimma* (semitone)	90	27
125:128	diesis	41	12.3
2025:2048	diaschima	20	6.
32768:32805	schisma	2	.6

Appendix III
Tone Numbers in the Marriage Allegory

The tone numbers in the final octave of the marriage allegory—6,480,000:12,960,000—constituting the fourth generation of Atlantis (cf. figs. 8 and 31) are given here together with their factoring into products of the prime numbers 2, 3, and 5. Numbers divisible by 60 = $2^2 3 \cdot 5$ also occur in the third generation octave 108,000:216,000. Numbers divisible by 60^2 occur in the second generation octave 1,800:3,600. Numbers divisible by 60^3 occur in the founders first octave 30:60. Notice that powers of 5 diminish by 1 in successive rows, while within each row powers of 3 increase as powers of 2 decrease. Had these numbers been written in Babylonian sexagesimal form they would have retained their smallest integer forms throughout the multiplications by 60; it is Greek decimal arithmetic which subjects the same meanings to so many arithmetical transformations.

The use of triple sharps (𝄪♯) and triple flats (♭♭♭) is a purely algebraic device to maintain the line of thought via musical thirds. Tones in rows 1 through 5 and 9 through 11 are 2 to 6 commas removed from the modern meanings of these letters, as can be seen from their loci in the tone circles of Atlantis (cf. fig. 30).

Row 1

1.	9,765,625 = 5^{10}	g𝄪♯

Row 2

1.	7,812,500 = $2^2 \cdot 5^9$	e𝄪
2.	11,718,750 = $2 \cdot 3 \cdot 5^9$	b𝄪

Row 3

1.	12,500,000 = $2^5 5^8$	c𝄪
2.	9,375,000 = $2^3 3 \cdot 5^8$	g𝄪
3.	7,031,250 = $2 \cdot 3^2 5^8$	d𝄪
4.	10,546,875 = $3^3 5^8$	a𝄪

Row 4

1.	10,000,000 = $2^7 5^7$	a♯
2.	7,500,000 = $2^5 3 \cdot 5^7$	e♯
3.	11,250,000 = $2^4 3^2 5^7$	b♯
4.	8,437,500 = $2^2 3^3 5^7$	f𝄪
5.	12,656,250 = $2 \cdot 3^4 5^7$	c𝄪

Row 5

1.	8,000,000 = $2^9 5^6$	f♯
2.	12,000,000 = $2^8 3 \cdot 5^6$	c♯
3.	9,000,000 = $2^6 3^2 5^6$	g♯
4.	6,750,000 = $2^4 3^3 5^6$	d♯

5. $10{,}125{,}000 = 2^3 3^4 5^6$ a♯
6. $7{,}593{,}750 = 2 \cdot 3^5 5^6$ e♯
7. $11{,}390{,}625 = 3^6 5^6$ b♯

Row 6

1. $12{,}800{,}000 = 2^{12} 5^5$ d
2. $9{,}600{,}000 = 2^{10} 3 \cdot 5^5$ a
3. $7{,}200{,}000 = 2^8 3^2 5^5$ e
4. $10{,}800{,}000 = 2^7 3^3 5^5$ b
5. $8{,}100{,}000 = 2^5 3^4 5^5$ f♯
6. $12{,}150{,}000 = 2^4 3^5 5^5$ c♯
7. $9{,}112{,}500 = 2^2 3^6 5^5$ g♯
8. $6{,}834{,}375 = 3^7 5^5$ d♯

Row 7

1. $10{,}240{,}000 = 2^{14} 5^4$ B♭
2. $7{,}680{,}000 = 2^{12} 3 \cdot 5^4$ F
3. $11{,}520{,}000 = 2^{11} 3^2 5^4$ C
4. $8{,}640{,}000 = 2^9 3^3 5^4$ G
5. $12{,}960{,}000 = 2^8 3^4 5^4$ D
6. $9{,}720{,}000 = 2^6 3^5 5^4$ A
7. $7{,}290{,}000 = 2^4 3^6 5^4$ E
8. $10{,}935{,}000 = 2^3 3^7 5^4$ B
9. $8{,}201{,}250 = 2 \cdot 3^8 5^4$ F♯
10. $12{,}301{,}875 = 3^9 5^4$ C♯

Row 8

1. $8{,}192{,}000 = 2^{16} 5^3$ g♭
2. $12{,}288{,}000 = 2^{15} 3 \cdot 5^3$ d♭
3. $9{,}216{,}000 = 2^{13} 3^2 5^3$ a♭
4. $6{,}912{,}000 = 2^{11} 3^3 5^3$ e♭
5. $10{,}368{,}000 = 2^{10} 3^4 5^3$ b♭
6. $7{,}776{,}000 = 2^8 3^5 5^3$ f
7. $11{,}664{,}000 = 2^7 4^6 5^3$ c
8. $8{,}748{,}000 = 2^5 3^7 5^3$ g
9. $6{,}561{,}000 = 2^3 3^8 5^3$ d
10. $9{,}841{,}500 = 2^2 3^9 5^3$ a
11. $7{,}381{,}125 = 3^{10} 5^3$ e

Row 9

1. $6{,}553{,}600 = 2^{18} 5^2$ e♭♭
2. $9{,}830{,}400 = 2^{17} 3 \cdot 5^2$ b♭♭
3. $7{,}372{,}800 = 2^{15} 3^2 5^2$ f♭
4. $11{,}059{,}200 = 2^{14} 3^3 5^2$ c♭
5. $8{,}294{,}400 = 2^{12} 3^4 5^2$ g♭
6. $12{,}441{,}600 = 2^{11} 3^5 5^2$ d♭
7. $9{,}331{,}200 = 2^9 3^6 5^2$ a♭
8. $6{,}998{,}400 = 2^7 3^7 5^2$ e♭
9. $10{,}497{,}600 = 2^6 3^8 5^2$ b♭
10. $7{,}873{,}200 = 2^4 3^9 5^2$ f
11. $11{,}809{,}800 = 2^3 3^{10} 5^2$ c
12. $8{,}857{,}350 = 2 \cdot 3^{11} 5^2$ g

Row 10

1. $10{,}485{,}760 = 2^{21} 5$ c♭♭
2. $7{,}864{,}320 = 2^{19} 3 \cdot 5$ g♭♭
3. $11{,}796{,}480 = 2^{18} 3^2 5$ d♭♭
4. $8{,}847{,}360 = 2^{16} 3^3 5$ a♭♭
5. $6{,}635{,}520 = 2^{14} 3^4 5$ e♭♭
6. $9{,}953{,}280 = 2^{13} 3^5 5$ b♭♭
7. $7{,}464{,}960 = 2^{11} 3^6 5$ f♭
8. $11{,}197{,}440 = 2^{10} 3^7 5$ c♭
9. $8{,}398{,}080 = 2^8 3^8 5$ g♭
10. $12{,}597{,}120 = 2^7 3^9 5$ d♭
11. $9{,}447{,}840 = 2^5 3^{10} 5$ a♭
12. $7{,}085{,}880 = 2^3 3^{11} 5$ e♭
13. $10{,}628{,}820 = 2^2 3^{12} 5$ b♭
14. $7{,}971{,}615 = 3^{13} 5$ f

Row 11

1. $8{,}388{,}608 = 2^{23}$ a♭♭♭
2. $12{,}582{,}912 = 2^{22} 3$ e♭♭♭
3. $9{,}437{,}184 = 2^{20} 3^2$ b♭♭♭
4. $7{,}077{,}888 = 2^{18} 3^3$ f♭♭
5. $10{,}616{,}832 = 2^{17} 3^4$ c♭♭
6. $7{,}962{,}624 = 2^{15} 3^5$ g♭♭
7. $11{,}943{,}936 = 2^{14} 3^6$ d♭♭
8. $8{,}957{,}952 = 2^{12} 3^7$ a♭♭
9. $6{,}718{,}464 = 2^{10} 3^8$ e♭♭
10. $10{,}077{,}696 = 2^9 3^9$ b♭♭
11. $7{,}558{,}272 = 2^7 3^{10}$ f♭
12. $11{,}337{,}408 = 2^6 3^{11}$ c♭
13. $8{,}503{,}056 = 2^4 3^{12}$ g♭
14. $12{,}754{,}584 = 2^3 3^{13}$ d♭
15. $9{,}565{,}938 = 2 \cdot 3^{14}$ a♭

Appendix IV
Introduction to the Monochord

Tuning theory is best demonstrated on the monochord, an instrument designed to permit the sounding length to be varied by a movable bridge without altering string tension, and of sufficient length to make commas and other micro-intervals both visible and audible.

The arithmetical complexities involved in defining the various tuning systems and the commas between them can be avoided by a simple paper-folding exercise analogous to the "rope-stretching" by which land was once measured and the proportions of buildings determined. The folding is done most conveniently on a strip of paper one or two inches wide and three to four feet long; adding machine tape serves very well. The examples described here were developed on a string and paper length of 120 centimeters. The pitches located at successive creases should be sounded on the string to reinforce the lesson for eye and ear.

The following exercise is carried through four stages.

1) Eight successive folding operations establish the diatonic major scale in Pythagorean tuning.
2) Five more folds establish the chromatic semitones needed for modal permutations, and a sixth fold reaches the Pythagorean comma 531441:524288.
3) Two folds establish the pure thirds of 5:4 needed for the Greek Dorian scale in Just tuning (related to the *Republic* scale), involving syntonic commas of 81:80; and two more establish the diesis 128:125.
4) The syntonic commas 81:80 are split visually to determine the approximate location of several tones in equal temperament.

Important insights are gained at each of the above four stages; failure to complete the whole series does not diminish satisfaction earned along the way. If the folding is done carefully, the errors involved will prove subliminal when tested on the string.

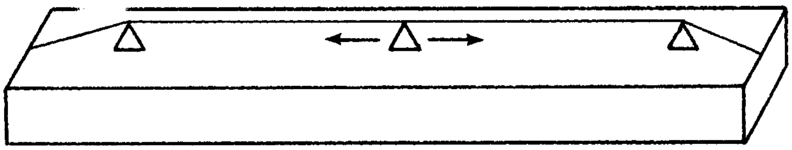


FIGURE 54

The Monochord

EXERCISE 1: THE C MAJOR SCALE IN PYTHAGOREAN TUNING

Cut the strip of paper exactly the same length as the sounding portion of the string. Mark the left end 0 (meaning zero string length) as a reference point for all folding operations, and mark the right end $^1/_1$= B as the schematic ground tone sounded by the full length of string. Later we shall dispense with this lowest tone, treating it as the Greeks treated the ground tone on their monochord, calling it *Proslambanomenos ("added* tone") and leaving it outside their tetrachord system. If folding always proceeds from 0 on the left, it will produce the descending octave scale on the right half of the paper strip. (If 0 were placed on the right and folding started from there, the same scale would appear ascending from the left.)

The first folding operation is the most important: it establishes the model for all that follows. Bring the two ends of the paper together and crease, then bring the doubled ends together and crease again. You have now divided the string into four parts: label them ¼ = B", ½ = B′, ¾ = E, making a vertical mark along each crease, whose locus will otherwise soon fade. Test on the string by sounding the successive lengths. Note the descending perfect fifth 2:3 at B′:E and the complementary perfect fourth 3:4 at E:B. Future folds are planned to project all tones into this octave.

Now consider these first results carefully to glean the two principles used from hereon:

1) To ascend a perfect fourth from any tone, equivalent to a multiplication by ¾, merely quarter its sounding length by folding in half twice (folding always from 0 towards the tone) and locating the new tone on the third crease (as the new tone E appeared here). The new tone appears on the left.

2) Alternatively, to descend a perfect fifth from any tone, equivalent to multiplication by $^3/2$, fold the reference length in half (folding always from 0 toward the tone length), and then use this folded portion to measure a third equal segment along the remainder. The new tone appears on the right, as E appears to the right of B′.

These two folding principles produce alternate higher fourths 3:4 and lower fifths 3:2 so that, like the piano tuner "laying the bearings" in the central octave, we keep all new tones within the compass of our chosen octave.

Now complete the heptatonic (7-tone) series by five more folds of the kind described above:

¾ E = A (reminder: divide O-E into fourths)

³⁄₂ A = *D (divide 0-A into halves, then measure along remainder)*

¾ D = G

³⁄₂ G = C

¾ C =F

The seven tones from B to B′ actually define the modern Locrian mode in Pythagorean tuning. Now halve C to produce C' and fold the end segment C:B back out of sight; the result is our familiar C major scale. Similar halving of successively higher tones would produce the Greek Phyrygian octave D-D′, the Greek Dorian octave E-E′, etc., but they teach us nothing new.

In Pythagorean tuning all major seconds are the same size, 8:9, slightly larger than the wholetones of equal temperament. The two minor seconds at E:F and B:C are undersized semitones of 243:256, which the Greeks called a *leimma* ("left-over," as the difference between two wholetones and a fourth 4:3). This is the tuning which modern string players approximate when they "stretch" intervals to intensify upward or downward "leading-tones." Pythagorean tuning remains relevant to modern harmony in the sense that the most powerful line of relationships extends through the dominantic order of perfect fourths and fifths. Such intervals are the strongest anchors for a mind seeking "shape" within the tonal flux.

EXERCISE 2: THE CHROMATIC SCALE AND PYTHAGOREAN COMMA

A continuation of the procedure outlined above will produce an endless number of alternating fourths and fifths. Five more folds will produce the five new tones needed for a twelve-tone chromatic scale in Pythagorean tuning.

¾ F = B♭ (reminder: divide 0-F into fourths).

³⁄₂ B♭ = E♭ (divide 0-B♭ into halves, then measure along remainder).

¾ E♭ = A♭

³⁄₂ A♭ = D♭

¾ D♭ =G♭

The chromatic scale shows alternate undersized diatonic semitone leimmas 243:256 (at C: B, B♭:A, A ♭: G, etc.,) and oversized chromatic semitone apotomes 2048:2187 (at B : B♭, A:A♭, G: G♭, etc.). One more folding operation produces a 13th tone (= 12th "disciple") which disagrees with the reference B' = ½ by a Pythagorean comma 531441:524288, an interval too small to allow the tones an independent status, yet large enough to produce an offensive disagreement which requires that one of them be eliminated from the set:

¾ G♭ = C♭, and C♭.B' is the Pythagorean comma.

A continuation of our tuning procedure would simply produce further commas with each of the original 12 tones in turn, hence we have reached the tonal boundary of Pythagorean tuning. (Note: If we allowed ourselves to make a more awkward triple fold , ≡ , of the original reference length B = $\frac{1}{1}$, we could locate at $\frac{2}{3}$= F♯ another Pythagorean comma G♭:F♯. Our procedure is designed to avoid these awkward, and perhaps logically inadmissible, triple folds.)

An appropriate selection from among the 12 tones produces the pattern of the ancient Greek Dorian mode, Plato's "true Hellenic mode," the pattern used in *Timaeus* for the World-Soul (in the C octave, not the D octave used in the text):

C		B♭		A♭		G		F		E♭		D♭		C
	t		t		s		t		t		t		s	

We notice here that the sequence of wholetones (t) and semitones (s) is exactly opposite to that of the C major scale.

EXERCISE 3: THE GREEK DORIAN SCALE IN JUST TUNING

By retuning two of the above tones we can produce the Greek Dorian scale in the so-called Didymus tuning, one of several forms of Just tuning, which always involves some mixture of fifths 2:3 and fourths 3:4 with pure thirds of 4:5. The folding procedure is a slight variant of that employed previously: quarter the reference length by a double fold, then measure a fifth segment at $\frac{5}{4}$ along the remainder.

$\frac{5}{4}$C′ = a♭ (starting always from 0, quarter the length for C′, to determine the measure for a♭, to its right).

$\frac{5}{4}$ G = e♭ (quarter the length for G, then measure e♭).

The micro-intervals a♭:A♭ and e♭:E♭ are syntonic commas 80:81. (They are the difference between a pure third $\frac{5}{4}$ = $\frac{80}{64}$ and a ditone third $(\frac{9}{8})^2 = \frac{81}{64}$.) The following tones and ratios are the Didymus tuning of the ancient Dorian octave.

C		B♭		a♭		G		F		E♭		d♭		C
	8:9		9:10		15:16		8:9		8:9		9:10		15:16	

The pure triad harmonies which became important to Western music in the 16th and 17th centuries required another variant of Just tuning, involving similarly warring commas in the tone field. Musicians were forced to disguise them by one subterfuge or another. What could not be disguised, however, were the worse disagreements which arose when two or three pure thirds 5:4 were taken in succession (as, for instance, in 19th c. Romantic harmonic modulations). Two more folding operations by 5:4 expose the *diesis* 125:128 which lies in wait; it is the discrepancy between three pure thirds 5:4 and the octave 2:1.

$\frac{5}{4}$ a♭ = *f*♭
$\frac{5}{4}$ *f*♭ = d♭♭

The ratio d♭♭ :C is $\frac{(5/4)^3}{2}$, or 125:128, the diesis.

EXERCISE 4: EQUAL TEMPERAMENT

Paper-folding cleverness cannot demonstrate the idea of a number like the “twelfth root of 2” (approximately 1.059463+) which provides the theoretical basis for an equal tempered scale. But because equal temperament tones lie *within* the syntonic commas we have already established, we can estimate the location of several of them with almost as much accuracy as the ear can appreciate. Interpolate the tempered A♭ within the syntonic comma a♭:A♭ of Just:Pythagorean tuning, the tempered E♭ within the syntonic comma e♭:E♭, and the tempered D♭ within the syntonic comma d♭:D♭, as illustrated in figure 55.

The ratio for an equal tempered semitone is one which carries us to the octave 2:1 after twelve identical operations. Vincenzo Galilei, famous

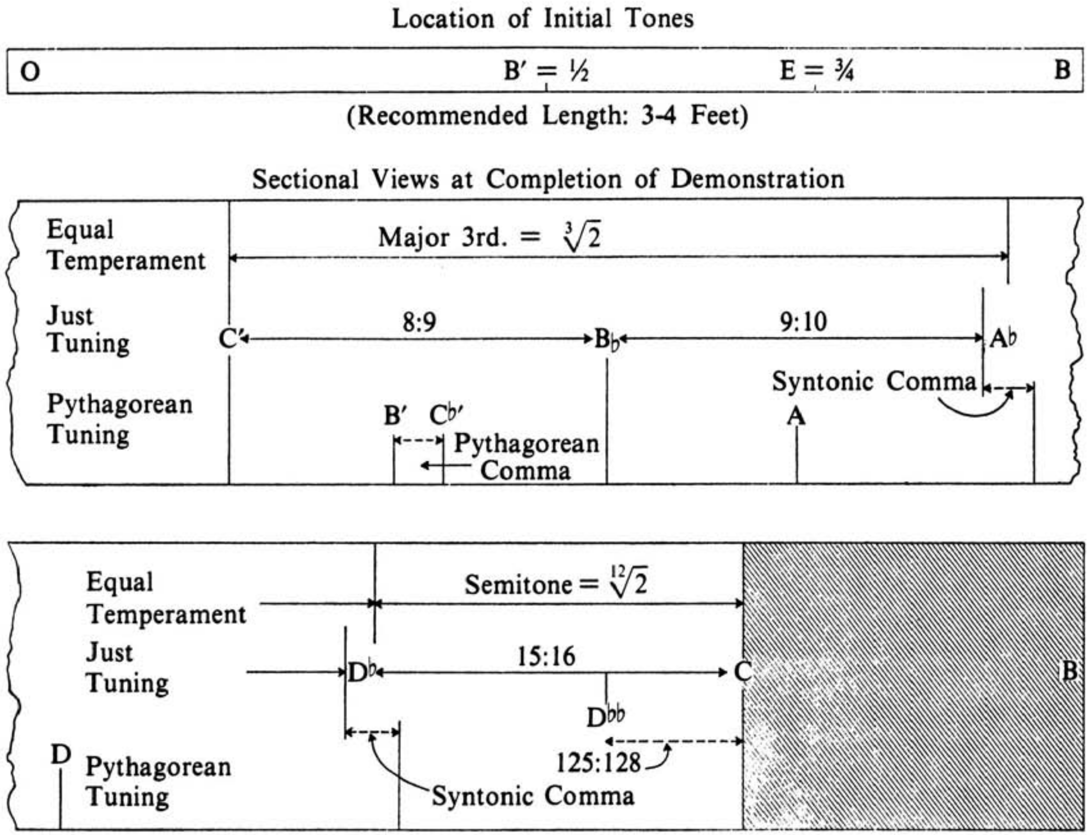


FIGURE 55

Pythagorean Tuning, Just Tuning, and Equal Temperament on the Monochord

father of an even more famous son, first explained how to achieve this to a sixteenth century lute-maker, relying on the ratio 18:17 an approximation at which he arrived not by calculation, we are told, but by intuition. Like Vincenzo we trust our intuition that the equal tempered tones we located in the middle of syntonic commas are accurate enough for present purposes that is, accurate enough to make clear that temperament is a compromise between the conflicting claims of various ideal values.

In tuning a scale we are haunted by the perfection which lies just beyond our grasp, and intrigued by the necessity for tempering even as much as lies within it. Paper-folding demonstrates the ear's problem to the eye: cyclic repetition at the octave requires that all smaller intervals be robbed of some measure of their perfection. None can be given "exactly what they are owed" if the system as a whole is to function at its best.

Footnotes

CHAPTER ONE

1. Thomas Heath, *Aristarchus of Samos* (London: Oxford University Press, 1913) pp. 94-115.

2. A. E. Taylor, *A Commentary On Plato's Timaeus* (London: Oxford University Press, 1928) p. 124.

3. James Adam, *The Republic of Plato* (Cambridge: University Press, 1902 and 1969) 2 vols.

4. Taylor, *op. cit.,* p. 96.

5. Francis M. Cornford, *Plato's Cosmology, The Timaeus of Plato* (New York: The Bobbs-Merrill Company, Inc. , reprint) p. 74.

6. *The Republic of Plato* (London: Oxford University Press, 1945) pp. 269 and 315.

7. Robert S. Brumbaugh, *Plato's Mathematical Imagination* (Bloomington: Indiana University Press, 1954; New York: Kraus Reprint, 1968).

8. *Ibid.,* p. 295.

9. Ibid., p. 88.

10. *Ibid.,* p. 226.

11. Albert von Thimus, *Die harmonikale Symbolik des Altherthums* 2 vols. (Köln, 1868-76, reprint Hildesheim, 1972).

12. Ernst Levy, *A Theory of Harmony* (Chicago: 1950, mimeographed).

____________The Pythagorean Tradition (University of Chicago, 1949-50, lecture notes, unpublished).

_____________“The Pythagorean Concept of Measure,” *Main Currents in Modern Thought,* vol. 21, no. 3, January—February 1965, pp. 51-57.

______________“A Response to 'The Threat and Promise of Cybernetics',” *Main Currents in Modern Thought,* vol. 22, no. 2, November—December 1965, pp. 48-50.

_______________with Siegmund Levarie, *Tone, A Study in Musical Acoustics*

(Kent: Kent State University Press, 1968).

_______________with Siegmund Levarie, *A Dictionary of Muscial*

Morphology, (Brooklyn: The Institute of Mediaeval Music, forthcoming).

13. Trevor J. Saunders, writing particularly of *Laws,* notes "a lot of elephantine punning and other kinds of word-play, usually impossible to reproduce in English," and describes Plato's Greek in that dialogue as "emphatic yet imprecise, elaborate yet careless, prolix yet curiously elliptical; the meaning is often obscure and the translator is forced to turn interpreter." See *Plato: The Laws* (Harmondsworth: Penguin Books Ltd., 1970) p. 39. Francis J. Cornford, in his translation of the *Republic,* notes that many key words, including 'music,' have shifted their meaning or "acquired false associations for English ears," and he chides Benjamin Jowett for translating literally Plato's statement that the best guide for a man's virtue is "philosophy tempered with music." See *The Republic of Plato* (London: Oxford University Press, 1945) p.vi. Allan Bloom, in attempting a more rigorously literal interpretation of the *Republic,* notes that Plato's dialogues "are constructed with an almost unbelievable care and subtlety," and flatly declares that "it is Socrates who rationalized music" (op. cit., pp. xviii and xiii). Bloom's own translation, together with his valuable footnotes and interpretative essay, was of considerable assistance in developing the musical interpretations being offered here.

14. Aristotle is explicit that Plato's curious "form-numbers" are limited to the first ten integers *(Metaphysics* 1084). Plato "expressly identified" numbers with the "Forms themselves" *(On The Soul,* 404). Aristotle insists that musical forms, on the contrary, are not numbers at all, but *ratios;* "the ratio 2:1 and number in general are causes of the octave" *(Metaphysics* 1013a). While Aristotle's understanding and reporting of Plato's metaphysics have been questioned, I find his criticism of Plato to be an infallible guide to the musical interpretations being offered here.

CHAPTER TWO

1. James Adam, *op. cit.,* vol. 2, p. 264.

2. Adam points out that "Plato was perfectly at liberty to call any number .. . which 'ends' or 'brings a consummation"' a perfect number *(ibid.,* p. 289). In an earlier essay I suggested that Plato intended factorial 6 = 720 to be understood ("Musical Marriages in Plato's *Republic," Journal of Music Theory,* vol. 18.2, Fall 1974, p. 248). Professor Sacksteder, however, suggests that it is simpler and more logical to suppose Plato intended 6, in the way shown here.

3. J. Murray Barbour, *Tuning and Temperament: A Historical Survey* (East Lansing: Michigan State College Press, 1953), p. 20.

4. A modern version of Ptolemy's tonal zodiac is presented in figure 48. Plato himself seems not to have known about the precession of the equinoxes (in a period slightly under 26,000 years). He declares that "the perfect number of time fulfills the perfect year" when the relative speeds of all eight revolutions of sun,

moon, planets and all-embracing celestial sphere "have accomplished their courses together" *(Timaeus* 39d). This fancy has little meaning for astronomers and had no arithmetical value for Plato, who declared that some of the planets had not yet been named or measured "one against another by numerical reckoning" *(Timaeus* 39c). To call the precessional period a "Platonic great year" does honor to a philosopher who believed celestial cycles *must* be perfectly coordinated; but his own notion of a "great" cycle requires several additional correlations and a period of time which he did not affect to understand.

5. See Thomas Heath, *op. cit.,* pp. 171-173, for a discussion of the tangled role 36,000 and 12,960,000 played in ancient astronomy.

6. A more serious comma develops between every one of the original eleven tones and the third tone higher or lower along the diagonals / / / etc. As can be seen in figure 6c, such tones have the ratio 125:64, falling short of an octave 128:64 by the ratio of the *diesis* 125:128, almost a quartertone. Here, I believe, is part of the rationale for Plato's "three increases," for once the ratio 4:5 is admitted into harmonic theory, it is of great interest to discover *how* it divides the octave. In Pythagorean perspective that means that we are interested in a conjunction" between powers of 2 which define octaves and powers of 5 which define thirds; $5^3 = 125$ is a Diophantine approximation to $2^7 = 128$; and once Socrates begins to toy with the "human number" 5, then three increases (meaning three consecutive thirds of 4:5) are required to fully demonstrate its deficiency.

CHAPTER THREE

1. James Adam, *op, cit.,* vol. 2, p. 358.

2. I am deeply indebted to James Adam *(loc. cit.,* p. 360) for making a line drawing—a pure monochord representation—of this arithmetical transformation. See figure 56, below.

CHAPTER FOUR

1. Robert S. Brumbaugh, *op. cit.,* p. 130.

2. The "plain of Truth" is referred to in *Phaedrus* (248b), a dialogue in which Zeus is described as leading the host of heaven "marshaled in eleven companies" (247a). Eleven tones are the most that can be tuned in the octave without encountering conflict. Hestia, wife of Zeus, must stay home to "mind the hearth," possibly because Platonic wives coincide with husbands as their octave doubles. The *Republic* is a dialogue between eleven men, named in the first few pages (but mainly carried on by Socrates and Plato's two older brothers, Glaucon and Adeimantus).

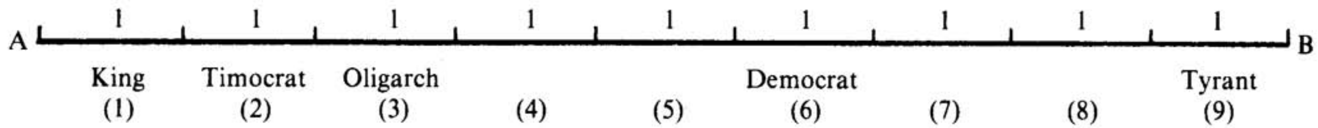


FIGURE 56
The Tyrant's Number

What is missing here are the reciprocals and the reduction to one model octave—easy enough if we obey Socrates' instructions to look for opposites and to study everything in cycles. By making a monochord drawing which left Plato's meaning still obscure,Adam encouraged further reflection. The "human number" 5—"in some ways like to its father, and in some ways like to its mother, being made up of three and two" (Plutarch)—is eliminated by this apparently whimsical renumbering, saving the planetary model and the World-Soul from several kinds of commas.

3. I read this as a dramatization of Brumbaugh's notion of an “octave modulus” (see footnote 10, chapter 1).

4. The Egyptian game with nested bowls, described in *Laws* (819), may have been devised to show how a cyclic “modular” arithmetic works.

5. See both Thomas Heath and James Adam, *op, cit.,* for extended discussions of correlations between Plato's structure and those of ancient astronomers. I have ignored entirely the possibility that he inherited part of his notions from Babylonian astronomy; in any case his arithmetic is Babylonian. But his procedure, I believe, is strictly musical.

6. B. L. van der Waerden, *Science Awakening* (New York: John Wiley & Sons, Inc. 1963) pp. 150-165.

CHAPTER FIVE

1. The tonal implications of Plato's “portions” can be read directly from the harmonic series: from any reference tone = 1 it progresses through the octave 2, the perfect fifth 3, the perfect fourth 4, etc.; and a reciprocal (falling) harmonic series, although not existing in nature, is readily imagined.

2. A. E. Taylor, *op. cit.* p. 137.

3. There is no problem concerning the means; Taylor, Cornford and Brumbaugh are in full agreement with ancient sources.

4. Thomas Taylor, trans., *The Commentaries of Proclus on the Timaeus of Plato,* 2 vol. (London: the author, 1820; Ann Arbor: University Microfilms 1969) pp. 57-73.

5. Except in *Laws,* which requires a three-dimensional graph, solutions to all Platonic musical allegories can be graphed as “Nicomachean” pebble patterns. The Latin word for “pebble” is *calculus.*

6. In modern notation, the arithmetic mean is $\frac{A+B}{2}$ and the harmonic mean is $\frac{2AB}{A+B}$

7. As Nichmachus states the rule, Every multiple will stand at the head of as many superparticular ratios corresponding in name with itself as it itself chances to be removed from unity, and no more nor less under any circumstances” *(Introduction to Arithmetic,* Bk. II, Ch. III,1). But the same is true of all tables built from numbers prime to each other, as can be seen from figures 7a and 8. Plato gives a similar clue in *Timaeus* (32b): “Solids are always conjoined, not by one mean, but by two.” Nicomachus points out that Plato requires that we be fluent in setting up “series of intervals of two, three, four, five, or an infinite number of . . . ratios” (Bk. II, Ch. 11,3). Nicomachus is translated by M. L. D'Ooge in vol. 11 of *Great Books of the Western World.*

8. A. E. Taylor, op. cit., p. 142.

9. Ibid., p. 111-112. Xenocrates saw with perfect clarity that Plato's Timaeus here in the creation myth is primarily concerned with "the logical derivation of the series of natural integers."

10. Aristotle's caustic commentary—"Of all the opinions we have enumerated, by far the most unreasonable is that which declares the soul to be a self-moving number" (On the Soul, 408b)—does not imply that he and Xenocrates are arguing about Plato's numbers. Xenocrates is describing how Plato uses numbers; Aristotle is concerned not with how but with why. We are grateful that their argument was loud enough to reverberate down through the centuries.

CHAPTER SIX

1. 1. Allan Bloom, *op. cit.,* p. 465, footnote 13.

CHAPTER SEVEN

1. For the development of some of the background materials, see Paul Friedlander, *Plato, An Introduction* (New York: Harper and Row, 1958) pp. 314-322.

2. Gilbert Ryle, *Plato's Progress* (Cambridge: Cambridge University Press, 1966) pp. 230-238.

3. My entire Atlantis interpretation is deeply indebted to Robert Brumbaugh's intensive study. As an example of his help, note his own map of the Atlantean Plain given below *(Plato's Mathematical Imagination,* fig. 19, p. 54).

4. Down through history Platonists treated *Critias* as a kind of fairy tale, but in 1882 a suddenly unemployed congressman from Minnesota named Ignatius J. Donnelly "devoted the large amount of time on his hands to assiduous study in the Library of Congress" and quickly produced *Atlantis: The Antediluvian World.* Since that date the flood of books and articles on Atlantis has steadily increased, and the end is nowhere in sight. The feverish activity in recent years actually to locate Plato's buried city off the coast of Greece, of Spain, or South America, or elsewhere, is a measure of naïveté. Donnelly's book, edited by Egerton Sykes, whom I quoted above, was revised in 1949 (New York: Gramercy Publishing Company). The adjective Atlantean, however, has

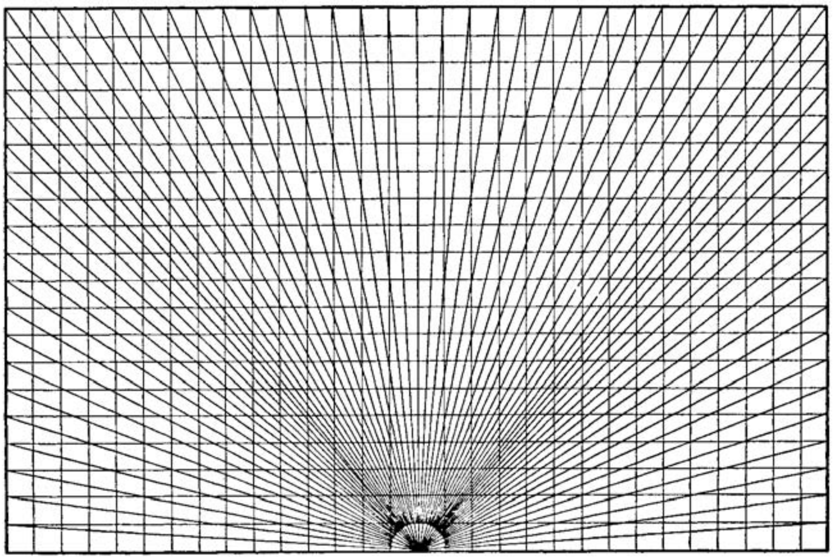


FIGURE 57

Canals in the Atlantean Plain

To correlate Brumbaugh's Plain with mine, consider every line in his to be a geometric progression, every intersection to be a number, remember that every number has two musical meanings (i.e., “two harvests”). My tonal diagrams are thus arithmetically redundant: upper and lower halves have the same meanings, and if one is eliminated then Brumbaugh's diagram remains, with the capitol city along one side of the plain, as Plato describes it. It requires a leap of pure musical imagination to move from his abstract logical diagrams to my explicitly arithmetical and tonal ones, and no one who feels timid about making that leap need be ashamed of his caution. My only regret is that every ancient mythology has not been studied in the precise way Brumbaugh has studied Plato, for only when scholars of his technical skill have prepared the ground thoroughly are musicians, like myself, likely to see the patterns of thought necessary to reconstruct plausible prototypes. Brumbaugh's advice and encouragement have been very important to me, and have led to many revisions of earlier analyses. In trying to let Plato tell his own story as succinctly as possible, I have failed to point out how very often my own path to musical understanding was via Brumbaugh's prior logical analysis.

become part of our modern vocabulary, as a general metaphor for all lost cities and civilizations, and in particular for gigantic statues which are properly objects of awe. The many prestigious scholars who have become involved in the search for Atlantis had no way of knowing, or even suspecting, that Plato's mathematical materials and literary procedures had not even been investigated by the musical methods he postulated.

5. 5. H. L. Jones, trans., *The Geography of Strabo* (Cambridge: Harvard University Press, 1949) 11-102 and XIII-598.

CHAPTER EIGHT

1. Trevor J. Saunders, *Plato: The Laws* (Harmondsworth: Penguin Books Ltd., 1970) p. 39. Saunders notes that "the reader of the *Republic* who picks up the *Laws* is likely to have difficulty in believing that the same person wrote both" (p. 27). I am indebted to Saunders not only for the lucidity of his translation and his helpful commentary but also for his comments on an earlier version of this chapter, helping me avoid a number of errors.

2. 2. Glenn R. Morrow, *Plato's Cretan City: A Historical Interpretation of the Laws* (Princeton: Princeton University Press, 1960) p. 196. Morrow's splendid work succeeds in tracing most of Plato's legislation to precedents in various ancient cities. His comment on music is interesting:

> *Moral order in the soul, and justice in the state, were both thought of in Platonic circles as analogous modes of attunement through the introduction of the required ratios and proportions. The use of mathematics in the latest period of Plato's thought was evidently much more extensive than anything revealed in the dialogues.*
>
> (p 506)

I have tried to show that Plato explicitly revealed virtually everything in these late dialogues, and that it is hidden only from those who never look for the musical implication he advertises.

CHAPTER NINE

1. Professor Sacksteder kindly computed these angles in hundredths of degrees to facilitate this presentation.

2. Mr. Robert Lawlor directed my attention to certain aspects of the Plimpton tablet which I had neglected in *The Myth of Invariance* (pp. 138-140), and thus prompted the discovery of the trigonometrical functions described here.

3. I owe this exposition to the French Egyptologist R. A. Schwaller de Lubicz, Whose magnum opus, *Le Temple de L' Homme* (Paris: Editions Caractères, 1957) Robert and Deborah Lawlor are now translating into English. De Lubicz exhibits a most remarkable empathy with Egyptian habits of thought, and believes that Babylonian and Egyptian mathematics were very closely related. His example of the Egyptian addition of equal angles suggests why the irrationality of certain lines in respect to others is relatively inconsequential when the primary mode of thinking is geometrical. Mr. Lawlor's kindness in acquainting me with the work of de Lubicz was the essential factor in helping me discover the trigonometric unity between the *Republic* and *Laws.*

4. "The division of the circumference of the circle into 360 parts originated in Babylonian astronomy of the last centuries B.C." (Neugebauer, *op. cit.* p. 25).

George Sarton *(op. cit., vol.* II, p. 287) discusses the roles of Hypsicles and Hipparchos in the 2nd c. B.C. Carl Boyer is inclined to give major credit for the systematic use of the 360° circle to Hipparchus *(op. cit.,* p. 180). The question raised here is whether or not Plato's formulas for a succession of angles varying by successive degrees, some of which are at least as old as Plimpton 322, argues for the conception of degrees at a far earlier period than we have supposed.

CHAPTER TEN

1. Harvey Wheeler, “The Invention of Political Theory” (1960, unpublished). This brilliant essay, kindly sent to me several years ago, rectifies many current misconceptions of Plato's political intentions.

2. Antonio T. de Nicolás, *Avatara: The Humanization of Philosophy Through the Bhagavad Gītā* (New York: Nicolas Hays, Ltd., 1976) pp. 282-299.
Meditations Through the Ṛg Veda: Four-Dimensional Man
(New York: Nicholas Hays Ltd., 1977).

3. Antonio T. de Nicolás, letter of January 16, 1977. It is interesting that arithmetic and geometry are both disciplines which encourage shifting viewpoints, that, for instance, the irrationals which provoked a crisis for the Pythagorean number theorist proved no problem at all for the geometer. The fusion of music, arithmetic, geometry and astronomy (the quadrivium of Archytas) provided the ancient philosopher with a highly articulated schema of changing perspectives, a richness threatened and finally lost by the conviction that “first principles” provide a superior platform of observation.

4. Robert S. Brumbaugh, “The Divided Line and the Direction of Inquiry,” *Philosophical Forum* II, 1944, p. 187.

5. Jacques Handschin, in his essay on “The *Timaeus* Scale” *(Musica Disciplina,* v. 4-5, 1950, pp. 3-42), the only essay on any Platonic mathematical allegory by a musicologist, so far as I know, begins by affirming that *Timaeus* disproves the current notion of Plato as “a thinker whose idealism takes a hostile view of the world of phenomena, despising beauty as perceived by the senses.” Plato's logical realism has obviously been over-emphasized by the Platonic tradition.

APPENDIX I

1. All Aristotle quotations are from *The Basic Works of Aristotle,* Richard McKeon, ed. (New York: Random House, 1941).

2. Ernest G. McClain, *The Myth of Invariance: The Origin of the Gods, Mathematics and Music from the Ṛg Veda to Plato* (New York: Nicolas Hays Ltd., 1976), p. 111.

3. Perhaps the most extensive study of Aristotle's criticism is that by Harold F. Cherniss in *Aristotle's Criticism of Plato and the Academy* (Baltimore: Johns Hopkins Pr., 1944; rpr. New York: Russell, 1962). This huge problem must be left for some future generation to tackle anew, at which time these musical interpretations should prove of some assistance.

4. A. E. Taylor, *op. cit.,* pp. 109-136.

5. Leo Schaya, *The Universal Meaning of the Kabbalah* (London: George Allen & Unwin Ltd. , 1971), pp. 1-60.

6. Eric Voegelin, *The Ecumenic Age* (Baton Rouge: Louisiana State University Press, 1974), p. 73.

7. A. E. Taylor, *op. cit.,* pp. 136-142.

8. Henry S. Macran, ed. and trans., *The Harmonics of Aristoxenus* (Oxford: The Clarendon Press, 1902).

9. B. L. van der Waerden, *op. cit.,* pp. 149-150.

10. Charles Davy, trans., *Section of the Canon,* in *Letters, Addressed Chiefly to a Young Gentleman* (Bury St. Edmunds, 1787), vol. 2, pp. 286-290.

11. Frank Cole Babbitt, trans., *Plutarch' s Moralia* V (Cambridge: Harvard University Press, 1936), pp. 134-137.

12. *Ibid.,* p. 175.

13. M. L. D'Ooge, trans., *Introduction to Arithmetic,* in vol. 11 of *Great Books of the Western World.*

14. Thomas L. Heath, *The Thirteen Books of Euclid's Elements,* 2nd ed., in 3 vols. (New York: Dover Publications, Inc., reprint 1956), vol. 2, p. 287.

15. George Sarton, *A History of Science,* 2 vols. (New York: W. W. Norton & Co. Inc. 1970, c. 1952 by Harvard College), vol. 1, pp. 205-208.

16. *Ibid.,* p. 207.

17. The most accessible presentation of von Thimus' table is in *Tone: A Stud in Musical Acoustics* by Siegmund Levarie and Ernst Levy (Kent: Kent State University Press, 1968), pp. 35-40.

18. There are so many contradictions and unanswered questions concerning the invention and use of zero that the truth is not likely to be known for some generations. Carl Boyer has discussed them extensively in *A History of Mathematics* (New York: John Wiley & Sons, Inc., 1968), pp. 234-244, and in "Zero: the Symbol, the Concept, the Number," *National Mathematics Magazine,* vol. 18, no. 8, May, 1944, pp. 1-8.

19. B. L. van der Waerden *(op. cit.,* p. 50) concludes that:

> *From remotest antiquity, the Greeks have known fractions and ... in the 5th century, at the latest, they had mastered the operations on fractions; reduction to lowest terms, to a common denominator, etc. For them therefore, in contrast with the Egyptians, difficulties in the calculation with fractions can not have been an obstacle in the way of the development of mathematics.*

20. Carl Boyer, A History of Mathematics, p. 235.

21. George Sarton, *op. cit., vol.* 1, p. 71, mentions the importance of Socrates' number in Babylon.

22. Ingemar Düring, "Ptolemaios Und Porphyrios Über Die Music," from Book III of *Ptolemy's Elements of Harmony, Goeteborgs Hoegskolas aarskrift,* 1934:1, p. 125.

23. My colleague Malcolm Brown brought Ptolemy's zodiac to my attention as the earliest example of a cyclic representation of the scale. Professor Brown also called my attention to Aristotle's "relaxed proportion." These ideas meet in Ptolemy's use of the ratio 9:8 for one sign of the zodiac, a value slightly too large unless "relaxed," as Aristotle suggests it might be.

24. J. Murray Barbour, *op. cit.,* pp. 15-24.

25. Walter Burkert discusses Archytas' tuning in *Lore and Science in Ancient Pythagoreanism* (Cambridge: Harvard University Press, 1972), pp. 387-389.

26. Carl Boyer, A *History of Mathematics,* p. 187.

27. One of the best introductions to ancient Greek musical theory is the article on "Music" by J. F. Mountford and R. P. Winnington-Ingram in *The Oxford Classical Dictionary,* 2nd ed. (Oxford: The Clarendon Press, 1970), pp. 705-712. Aristides is discussed on p. 709. Another excellent introduction is the article on "Greek Music (Ancient)" by Winnington-Ingram in *Grove's Dictionary of Music and Musicians,* 5th ed. (New York: St. Martin's Press, 1954), vol. 3. Aristides' tunings are given on p. 777.

28. George Sarton, *op. cit.,* p. 400.

29. A. E. Taylor, *op. cit.,* discusses the *Timaeus Locris* in detail, pp. 37 and 655-664.

30. Carl Boyer, A *History of Mathematics,* pp. 97-98.

31. There are two convenient sources for the following discussion: 1) Walter Burkert, *op. cit.,* pp. 394-400, and 2) John Hawkins, A *General History of the Science and Practice of Music,* 2 vols. (New York: Dover Publications, Inc., 1963, unabridged republication of J. Alfred Novello edition of 1853), vol. 1, pp. 27-28 and 123.

32. The Philolaus comma in figure 49, if it is fairly represented by the ratio 4374:4330, is worth only about 17 cents, hence there would be approximately 12 of them in a wholetone 9:8 worth 204 cents (12 × 17 = 204). Yet he gives the correct answer, 9—correct, that is, for the only comma the Greeks could have calculated accurately and easily, 80:81.

The Greeks of Plato's century are not credited with an interest in large numbers, yet Xenocrates is reputed to have calculated the number of syllables that could be formed with the letters of the Greek alphabet as 1,002,000,000,000 (Sarton, *op. cit., vol.* 1, p. 503).

33. I owe the title, *The Pythagorean Plato,* to Professor Francis D. Wormuth of the University of Utah, who suggested it during the course of a long and valuable correspondence.

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There is something very special about this book as regards both method and contents. Scholars in various fields—mathematics, philology, political science, education, philosophy, music—can profit from it significantly and in due time are likely to recognize it as essential to further thought in their respective disciplines.

Plato's dialogues abound in musical examples and their corresponding mathematical equations which have driven philologists and philosophers of the last two millennia to open despair. Ernest G. McClain, a music professor [emeritus] at the City University of New York, has succeeded where angels either feared to tread or tripped. His method is simple enough: he started by taking Plato literally. He believed Plato's statements concerning the supremacy of music: "Education in music is most sovereign" (*Republic* 501d). "Argument mixed with music... alone, when it is present, dwells within the one possessing it as a savior of the virtue throughout life" (*Republic* 549b). "Moderation" will stretch through the whole city as it does "from top to bottom of the entire scale, making the weaker, the stronger, and those in the middle... sing the same chant together" (*Republic* 432a). "Our songs have become laws" (*Laws* 799d).

McClain's method, in short consists in fully respecting Plato's insistence on the musical interpretation of numbers, and in accepting and translating the somewhat clumsy Greek mathematical formulations. The results, all of them intelligible to thinking musicians, are highly persuasive. The contents emerge as various attempts at coordinating disparate elements to function in an orderly manner within a system. Plato's late dialogues form a continuity extending from the *Republic* through *Timaeus* and *Critias* to *Laws*. In the face of Plato scholars who generally hesitate to elucidate one dialogue by another, McClain has had the courage to trace the musical connections [using] the ingenious idea of interpreting Plato's three different relevant number sets as different kinds of musical temperament. [His] analysis could not be more specific. He supports every step in his logical chain by direct references to Plato's own words. McClain's accomplishment strikes me as a major intellectual breakthrough in man's perennial struggle with his own spiritual history.

Siegmund Levarie
from a review in the July, 1978 issue
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